THE ITERATED STIELTJES TRANSFORM*

BY

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Introduction

This paper is concerned with the iterate of the Stieltjes transform

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}, \tag{1}$$

which in turn is the iterate of the Laplace transform

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t). \tag{2}$$

It is known† that (2) can be inverted by use of the differential operator of infinite order

$$\lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} f^{(k)}(k/t),$$

an operator which annuls the functions

$$f(x) = x^k, \qquad k = 0, 1, 2, \cdots. \tag{3}$$

It is also known‡ that (1) can be inverted by use of the linear differential operator of infinite order

$$\lim_{k\to\infty} \frac{(-t)^{k-1}}{k!(k-2)!} \left[t^k f(t)\right]^{(2k-1)},$$

an operator which annuls the functions (3), where now k runs through the negative integers as well.

When the transform (1) is iterated, one is led to the transform

$$f(x) = \int_0^\infty \frac{du}{x+u}\int_0^\infty \frac{d\alpha(t)}{u+t}, \tag{4}$$

or, when it is permissible to change the order of integration, to the transform

$$f(x) = \int_0^\infty \frac{\log(x/t)}{x-t}\, d\alpha(t). \tag{5}$$

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† D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, these Transactions, vol. 36 (1934), p. 107.

‡ See the paper of D. V. Widder cited in §3.

To distinguish between these two cases we refer to (4) as the *iterated Stieltjes transform* and to (5) as the S_2 *transform*. We show that the existence of the integral (5) implies the existence of the repeated integral (4), but not conversely.

It should be observed that the kernel

$$(x-t)^{-1}\log(x/t)$$

becomes infinite as t approaches zero. For this reason the integral (5) must be understood to mean

$$\lim_{\epsilon\to 0,\, R\to\infty}\int_\epsilon^R \frac{\log(x/t)}{x-t}\,d\alpha(t).$$

When it is desirable to emphasize that this Cauchy value of the integral is intended, we write it as

$$\int_{0+}^\infty \frac{\log(x/t)}{x-t}\,d\alpha(t).$$

In the first part of the present paper the inversion of the integrals (4) and (5) is discussed. It is found that the inversion operator is again a linear differential operator which annuls the functions (3) for $k=0, \pm 1, \pm 2, \cdots$ and in addition the functions

$$f(x) = x^k \log x, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

For k an integer greater than unity we define an operator $H_{k,t}[f]$ by the relation

$$H_{k,t}[f(x)] = \left[\frac{1}{k!(k-2)!}\right]^2 \left\{t^{2k-1}\left[t^{2k-1}f^{(k-1)}(t)\right]^{(2k-1)}\right\}^{(k)}.$$

We are then able to show that if $\alpha(t)$ is an integral and is of such a nature that (4) or (5) exists, then

$$\lim_{k\to\infty} H_{k,t}[f(x)] = \alpha'(t)$$

for almost all positive values of t. If $\alpha(t)$ is of bounded variation in every finite interval and is of such a nature that (4) or (5) exists, then

$$\tfrac{1}{2}[\alpha(t+)+\alpha(t-)] = \alpha(0+) + \lim_{k\to\infty}\int_{0+}^t H_{k,u}[f(x)]\,du$$

for all positive values of t.

In the remaining part of the paper necessary and sufficient conditions for the representation of functions in the forms (4) and (5) are discussed. The most important results are summarized in the following table.

Class of the function $\alpha(t)$	*Condition*
(A) Non-decreasing	$H_{k,t}[f] \geqq 0$, $(t>0)$
(B) Of bounded variation on $(0, \infty)$	$\int_0^\infty \lvert H_{k,t}[f]\rvert dt \leqq M$
(C) Integral of a function of L^p, $(p>1)$	$\int_0^\infty \lvert H_{k,t}[f]\rvert^p dt \leqq M$
(D) Integral of a function of L	$\text{l.i.m.}^{(1)}_{k\to\infty} H_{k,t}[f]$ exists
(E) Integral of a bounded function	$\lvert H_{k,t}[f]\rvert \leqq M$, $(t>0)$

$$\begin{aligned} f(x) &= o(x^{-1}), && x \to 0+, \\ f(x) &= o(1), && x \to \infty. \end{aligned} \tag{6}$$

An entry in the right-hand column of this table indicates that those conditions for an infinite sequence of positive integers k plus conditions (6) are necessary and sufficient for the representation of $f(x)$ in the form (4) with $\alpha(t)$ a function of the class described in the corresponding left-hand column. It is found that an additional condition must be added, except in cases (A) and (C), for representation in the form (5).

The method of proof is such that from the conditions in the right-hand column one must be able to infer that

$$\begin{aligned} f^{(k)}(x) &= o(x^{-k-1}), && x \to 0+, \\ f^{(k)}(x) &= o(x^{-k}), && x \to \infty, \end{aligned}$$

for all non-negative integers k. This is done by use of a result of R. P. Boas* concerning the asymptotic behavior of Euler differential forms.

The proofs of our representation theorems are necessarily complicated by the fact, observed above, that the kernel of equation (5) is not bounded. A bounded auxiliary kernel

$$E(x, t) = Q_x\left[\frac{\log (x/t)}{x - t}\right] = 2x\int_0^\infty \frac{u^2 du}{(x + u)^3(t + u)^2},$$

where

$$Q[f(x)] = x[x^2 f'(x)]'',$$

is used. Conditions for the representation of $f(x)$ in the form

$$f(x) = \int_0^\infty E(x, t) d\alpha(t)$$

* See the reference in §14.

are first obtained, and the transition to the form (4) is then made by use of a Tauberian theorem.

The two inversion formulas, for S_2 transforms, and the representation theorem (C) have been previously announced by D. V. Widder.*

Chapter I. Properties of the transforms

1. **The S_2 transform.** Let $\alpha(u)$ be a function, defined on $(0, \infty)$, of bounded variation on every interval (ϵ, R), $(0<\epsilon<R<\infty)$, and normalized by the conditions

$$\alpha(0) = 0, \qquad \alpha(u) = \tfrac{1}{2}[\alpha(u+) + \alpha(u-)], \qquad u > 0. \tag{1.1}$$

For $x>0$ we consider the limit

$$f(x) = \lim_{\epsilon\to 0+,\, R\to\infty} \int_\epsilon^R \frac{\log(x/t)}{x-t}\, d\alpha(t), \tag{1.2}$$

where $(\log x - \log t)/(x-t)$ is defined by its limiting value $1/x$, for $t=x$.

Definition 1.1. *For any points $x>0$ for which the limit* (1.2) *exists, defining a function $f(x)$, $f(x)$ is said to be an S_2 transform, convergent for such points. We write*

$$f(x) = \int_{0+}^\infty \frac{\log(x/t)}{x-t}\, d\alpha(t). \tag{1.3}$$

The function $\alpha(t)$ is called the determining function of $f(x)$.

Theorem 1.1. *If the S_2 transform* (1.3) *converges for some $x_0>0$, it converges for every $x>0$ and converges uniformly in any interval $a\leqq x\leqq A$, where $0<a<A<\infty$.*

It is necessary to show that the two integrals

$$\int_{2A}^\infty \frac{\log(x/t)}{x-t}\, d\alpha(t), \qquad \int_{0+}^{a/2} \frac{\log(x/t)}{x-t}\, d\alpha(t) \tag{1.4}$$

converge uniformly, $a\leqq x\leqq A$.

To discuss the first integral (1.4), we set

$$\beta(t) = \int_B^t \frac{\log(x_0/u)}{x_0-u}\, d\alpha(u), \qquad \beta(B) = 0,$$

where $B>2A$ is a sufficiently large constant. By hypothesis, $\beta(\infty)$ exists. For $a\leqq x\leqq A$ and $R>B$,

* D. V. Widder, *The iterated Stieltjes transform*, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 242–244.

$$\int_B^R \frac{\log (x/t)}{x-t}\, d\alpha(t) = \int_B^R \frac{\log x - \log t}{\log x_0 - \log t}\, \frac{x_0 - t}{x-t}\, d\beta(t)$$

$$= \beta(R) \frac{\log x - \log R}{\log x_0 - \log R}\, \frac{x_0 - R}{x - R}$$

$$- \log \frac{x}{x_0} \int_B^R \beta(t)\, \frac{x_0 - t}{x - t}\, \frac{dt}{t(\log x_0 - \log t)^2}$$

$$- (x_0 - x) \int_B^R \beta(t)\, \frac{\log x - \log t}{\log x_0 - \log t}\, \frac{dt}{(x-t)^2}$$

$$= J_1 - J_2 - J_3.$$

Now let $R \to \infty$. The term J_1 is $\beta(R)$ multiplied by a bounded factor which approaches unity, so that $\lim_{R\to\infty} J_1 = \beta(\infty)$, uniformly for $a \leqq x \leqq A$. Since $\beta(t)$ is bounded, it is easily seen that J_2 is dominated by an integral of the form

$$\int^\infty \frac{M dt}{t(C + \log t)^2},$$

where M and C are constants, independent of x, and that J_3 is dominated by an integral of the form

$$\int^\infty \frac{M dt}{(C + t)^2}.$$

Thus J_2 and J_3 approach limits as $R \to \infty$, uniformly for $a \leqq x \leqq A$. To treat the other integral (1.4), we consider

$$\lim_{\epsilon \to 0+} \int_\epsilon^{a/2} \frac{\log (x/t)}{x - t}\, d\alpha(t), \tag{1.5}$$

and set $x = y^{-1}$, $t = u^{-1}$, and $\epsilon = R^{-1}$; the limit (1.5) becomes

$$\lim_{R\to\infty} y \int_{2/a}^R \frac{\log (y/u)}{y - u}\, d\gamma(u), \qquad \gamma(u) = \int_{2/a}^u (-v) d\alpha(v^{-1}),$$

which is a limit of the form already discussed.

We have defined the S_2 transform only for a real variable x. If we regard x as a complex variable in (1.2) and admit all determinations of the logarithmic function, we may still call the function defined by (1.2) an S_2 transform. In this paper we shall not discuss the S_2 transform in the complex domain; but we state here, without proof, some of its properties.

THEOREM 1.2. *If an S_2 transform converges for any complex $x \neq 0$, with any determination of the logarithm, it converges for every x not on D, the positive real axis, with any determination of the logarithm, and for x on D if the principal value* of the logarithm is used. If, in a region S in which the S_2 transform converges, the determinations of* $\log x$ *used form an analytic function, then the S_2 transform represents a function analytic in S. The analytic function obtained by using the principal value of* $\log x$ *and continuing the result analytically has $x=0$ as a singular point.*

2. **Lemmas on Stieltjes integrals.** We prove the following lemma:

LEMMA 2.1. *Let $f(x)$ be bounded, nonnegative, and monotonic on the (finite) interval $a \leqq x \leqq b$. Let $\alpha(x)$ have bounded variation on $a \leqq x \leqq b$. Let $\int_a^b f(x)d\alpha(x)$ exist. Then, according as $f(x)$ is non-decreasing or non-increasing,*

$$f(b) \underset{a \leqq x \leqq b}{\text{l.b.}} [\alpha(b) - \alpha(x)] \leqq \int_a^b f(x)d\alpha(x) \leqq f(b) \underset{a \leqq x \leqq b}{\text{u.b.}} [\alpha(b) - \alpha(x)],$$

or

$$f(a) \underset{a \leqq x \leqq b}{\text{l.b.}} [\alpha(x) - \alpha(a)] \leqq \int_a^b f(x)d\alpha(x) \leqq f(a) \underset{a \leqq x \leqq b}{\text{u.b.}} [\alpha(x) - \alpha(a)].$$

This lemma will usually appear in the form which it assumes when $\alpha(x) = \int_a^x \phi(t)d\beta(t)$, where $\beta(t)$ is a function of bounded variation on $a \leqq x \leqq b$, and $\phi(t)$ is a bounded function such that $\alpha(x)$ is defined. If $\alpha(x)$ is a Lebesgue integral, the lemma reduces to "Bonnet's theorem,"† since (taking for definiteness the case where $f(x)$ is non-decreasing)

$$\alpha(x) - \alpha(a) = \int_a^x \alpha'(t)dt,$$

and since the continuous function

$$f(b) \int_a^x \alpha'(t)dt$$

takes on every value between its maximum and minimum, and, in particular, the value

$$\int_a^b f(t)\alpha'(t)dt.$$

* $-\pi < \Im[\log x] \leqq \pi$, where the symbol $\Im$ denotes "imaginary part of."

† See, for example, E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, vol. 1, 1927, p. 618.

The two cases of the lemma are equivalent, by the substitution $-y=x$. We consider the case where $f(x)$ is non-decreasing. Then

$$\begin{aligned}\int_a^b f(x)d\alpha(x) &= -\int_a^b f(x)d[\alpha(b)-\alpha(x)]\\ &= f(a)[\alpha(b)-\alpha(a)]+\int_a^b [\alpha(b)-\alpha(x)]df(x)\\ &\leqq f(a)\underset{a\leqq x\leqq b}{\text{u.b.}}[\alpha(b)-\alpha(x)]+\underset{a\leqq x\leqq b}{\text{u.b.}}[\alpha(b)-\alpha(x)]\int_a^b df(x)\\ &= f(b)\underset{a\leqq x\leqq b}{\text{u.b.}}[\alpha(b)-\alpha(x)].\end{aligned}$$

The inequality in the other sense is established similarly.

Lemma 2.2. *Let $f(x)$ be bounded, nonnegative, and monotonic on the (finite or infinite) interval (a, b). Let $\alpha(x)$ have bounded variation on $(a+\epsilon, b-\epsilon)$* for every (sufficiently small) $\epsilon>0$. Let A and B mean, respectively, either a or $a+$, b or $b-$. Then if $\alpha(b-)$ and $\int_A^{b-}f(x)d\alpha(x)$ exist, and if $f(x)$ is non-decreasing, then*

$$f(b-)\underset{a\leqq x\leqq b}{\text{l.b.}}[\alpha(b-)-\alpha(x)]\leqq\int_A^{b-}f(x)d\alpha(x)\leqq f(b-)\underset{a\leqq x\leqq b}{\text{u.b.}}[\alpha(b-)-\alpha(x)];$$

if $\alpha(a+)$ and $\int_{a+}^{B}f(x)d\alpha(x)$ exist, and if $f(x)$ is non-increasing, then

$$f(a+)\underset{a\leqq x\leqq b}{\text{l.b.}}[\alpha(x)-\alpha(a+)]\leqq\int_{a+}^{B}f(x)d\alpha(x)\leqq f(a+)\underset{a\leqq x\leqq b}{\text{u.b.}}[\alpha(x)-\alpha(a+)].$$

Let us consider the case $b<\infty$, $A=a$; the details for the other cases are similar and may be left to the reader. We determine, for $\epsilon>0$, a function $\delta=\delta(\epsilon)>0$ such that

$$|\alpha(b-)-\alpha(b-\delta)|=\left|\int_{b-\delta}^{b-}d\alpha(t)\right|\leqq\epsilon,\qquad \lim_{\epsilon\to 0}\delta(\epsilon)=0.$$

Then by use of Lemma 2.1,

$$\begin{aligned}\int_a^{b-\delta}f(t)d\alpha(t) &\geqq f(b-\delta)\underset{a\leqq x\leqq b-\delta}{\text{l.b.}}\int_x^{b-\delta}d\alpha(t)\\ &\geqq f(b-\delta)\underset{a\leqq x\leqq b}{\text{l.b.}}\left[\int_x^{b-}d\alpha(t)-\int_{b-\delta}^{b-}d\alpha(t)\right]\\ &\geqq f(b-\delta)\underset{a\leqq x\leqq b}{\text{l.b.}}\int_x^{b-}d\alpha(t)-\epsilon f(b-\delta).\end{aligned}$$

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* If $a=-\infty$, $a+\epsilon$ means $-\epsilon^{-1}$; if $b=+\infty$, $b-\epsilon$ means ϵ^{-1}.

Let $\epsilon \to 0$; then we obtain our inequality in one sense. The opposite inequality is obtained similarly.

LEMMA 2.3. *Let $\phi(t)$ have bounded variation on $a \leqq x \leqq b$ for every $b > a$; let $\phi(\infty)$ exist. Then if $\int_a^b \psi(t)d\phi(t)$ exists for every $b > a$, and if $\psi(t)$ is, for t greater than some t_0, non-negative, monotonic, and bounded, the integral $\int_a^\infty \psi(t)d\phi(t)$ converges. If $\psi(t)$ depends on a parameter, and if t_0 and the bound for $\psi(t)$ are independent of the parameter, the convergence is uniform with respect to the parameter.*

In the applications which we shall make, the lemma will usually occur with $\phi(t)$ an integral. When $\phi(t)$ is a step-function, the lemma reduces to "Abel's test" for infinite series.*

The lemma is a simple consequence of Lemma 2.2. Assume $0 \leqq \psi(t) < B, (t > t_0)$. Given $\epsilon > 0$, choose T so large that for $T'' > T' \geqq T$,

$$| \phi(T'') - \phi(T') | < \epsilon B^{-1}.$$

Take $S' > S > T$. Then

$$\left| \int_S^{S'} \psi(t)d\phi(t) \right| \leqq \begin{cases} \psi(S') \underset{S \leqq S'' \leqq S'}{\text{u.b.}} | \phi(S') - \phi(S'') |, \\ \psi(S) \underset{S \leqq S'' \leqq S'}{\text{u.b.}} | \phi(S'') - \phi(S) |, \end{cases}$$

according as $\psi(t)$ is non-decreasing or non-increasing, respectively; and

$$\left| \int_S^{S'} \psi(t)d\phi(t) \right| < \epsilon, \qquad S' > S > T.$$

This establishes the stated convergence.

LEMMA 2.4. *Let $\phi(t^{-1}), \psi(t^{-1})$ satisfy the conditions of Lemma 2.3 with $a > 0$. Then the integral $\int_{0+}^{1/a} \psi(t)d\phi(t)$ exists.*

This is reduced to Lemma 2.3 by the change of variable $t = u^{-1}$.

One simple application of Lemmas 2.3 and 2.4 is worth stating separately.

LEMMA 2.5. *If the S_2 transform* (1.3) *converges, the integrals*

$$(2.1) \qquad \int_{0+}^{1/2} \frac{d\alpha(t)}{1-t}, \qquad \int_2^\infty \frac{d\alpha(t)}{1-t}$$

converge.

If (1.3) converges, it converges for $x = 1$. Since the functions $-1/\log t$ and $1/\log t$ are positive, monotonic, and bounded on $(0, 1/2)$ and on $(2, \infty)$, re-

* K. Knopp, *Theory and Application of Infinite Series*, 1928, p. 314.

spectively, the convergence of (2.1) follows by Lemmas 2.4 and 2.3 from the convergence of

$$\int_{0+}^{\infty} \frac{\log t}{1-t}\, d\alpha(t).$$

3. **The determining function of an S_2 transform.** We prove the following theorem:

THEOREM 3.1. *If* (1.3) *converges, then* $\alpha(0+)$ *exists, and*

$$\alpha(t) - \alpha(0+) = o(-1/\log t), \qquad t \to 0, \tag{3.1}$$

$$\int_t^{\infty} u^{-1} d\alpha(u) = o(1/\log t), \qquad t \to \infty; \tag{3.2}$$

$$\alpha(t) = o(t/\log t), \qquad t \to \infty. \tag{3.3}$$

From Theorem 1.1 and Lemma 2.5 we see that

$$\int_{0+}^{1/2} \frac{\log t}{1-t}\, d\alpha(t), \quad \int_{2}^{\infty} \frac{\log t}{1-t}\, d\alpha(t), \quad \int_{0+}^{1/2} \frac{d\alpha(t)}{1-t}, \quad \int_{2}^{\infty} \frac{d\alpha(t)}{1-t}$$

converge; a simple application of Lemmas 2.3 and 2.4 then shows that

$$\int_{0+}^{1} d\alpha(t), \tag{3.4}$$

$$\int_{0+}^{1} \log t\, d\alpha(t), \tag{3.5}$$

$$\int_{1}^{\infty} t^{-1} \log t\, d\alpha(t), \tag{3.6}$$

$$\int_{1}^{\infty} t^{-1} d\alpha(t) \tag{3.7}$$

exist. The existence of (3.4) implies the existence of $\alpha(0+)$. If we then write

$$\alpha(t) - \alpha(0+) = \int_{0+}^{t} d\alpha(u) = \int_{0+}^{t} \frac{d\beta(u)}{\log u}, \quad \beta(t) = \int_{0+}^{t} \log u\, d\alpha(u),$$

we have, because (3.5) converges,

$$\beta(t) = o(1), \qquad t \to 0,$$

$$\alpha(t) - \alpha(0+) = \frac{\beta(t)}{\log t} - \int_{0+}^{t} \beta(u) d(1/\log u), \qquad 0 < t < 1,$$

$$= o(-1/\log t), \qquad t \to 0.$$

This is (3.1).

Since (3.7) converges,

$$\int_t^\infty t^{-1}d\alpha(t) = \int_t^\infty \frac{1}{\log u}\, d\beta(u), \qquad \beta(t) = \int_t^\infty u^{-1}\log u\, d\alpha(u).$$

The convergence of (3.6) implies that $\beta(\infty)=0$; hence

$$\begin{aligned}\int_t^\infty t^{-1}d\alpha(t) &= \frac{\beta(t)}{\log t} - \int_t^\infty \beta(u)d(1/\log u), && t > 1,\\ &= o(1/\log t), && t \to \infty.\end{aligned}$$

This is (3.2), and (3.3) may be obtained from it; or we may proceed as follows. Because

$$\int_{0+}^\infty \frac{\log t\, d\alpha(t)}{1+t}$$

converges,

$$\beta(t) = \int_2^t \log u\, d\alpha(u) = o(t), \qquad t \to \infty.*$$

Then with $\beta(2)=0$,

$$\begin{aligned}\alpha(t) - \alpha(2) &= \int_2^t \frac{\log u\, d\alpha(u)}{\log u} = \int_2^t \frac{d\beta(u)}{\log u}\\ &= \frac{\beta(t)}{\log t} - \int_2^t \beta(u)d(1/\log u) = o(t/\log t), \qquad t \to \infty.\end{aligned}$$

4. **Properties of the Stieltjes transform.** The Stieltjes transform in its usual form

(4.1)
$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}$$

assumes $\alpha(t)$ of bounded variation in $0 \leqq t \leqq R$ for every positive R. We shall need to consider also the transform

(4.2)
$$f(x) = \int_{0+}^\infty \frac{d\alpha(t)}{x+t} = \lim_{\epsilon\to 0+,\, R\to\infty} \int_\epsilon^R \frac{d\alpha(t)}{x+t},$$

where $\alpha(t)$ is of bounded variation in (ϵ, R) if only $0<\epsilon<R<\infty$. For example, if $\alpha(t)=t\sin(t^{-1})$ when $0<t<1$, and if $\alpha(t)=0$ when $t=0$ and when $1\leqq t \leqq \infty$, then (4.2) exists although (4.1) is undefined. On the other hand $f(x)=x^{-1}$ can have the representation (4.1) but not (4.2).

* D. V. Widder, *The Stieltjes transform*, these Transactions, vol. 43 (1938), pp. 7–60.

By an obvious change of variable we have

$$\int_{0+}^{\infty} \frac{d\alpha(t)}{x+t} = \int_{1}^{\infty} \frac{d\alpha(t)}{x+t} - \frac{1}{x}\int_{1}^{\infty} \frac{t\,d\alpha(1/t)}{t+x^{-1}}.$$

The first integral on the right is in the form (4.1); the second is also except that x has been replaced by its reciprocal. This enables us to derive easily the facts we need concerning (4.2) from the known results about (4.1).* In particular we showed in §3 that the convergence of (4.2) at $x=1$ implies the existence of $\alpha(0+)$.

We summarize what we shall need in the following theorem:

THEOREM 4.1. *If* (4.2) *converges for some* $x_0>0$, *it converges for every* $x>0$, *and converges uniformly on any interval* $R\geqq x\geqq\delta$, $(0<\delta<R<\infty)$; $f(x)$ *is analytic for* $x>0$, *and its derivatives may be evaluated by Leibniz' rule*; *furthermore,*

$$\alpha(0+) \text{ exists}; \tag{4.3}$$

$$\alpha(t) = o(t), \qquad t\to\infty; \tag{4.4}$$

$$f^{(n)}(x) = o(x^{-n-1}), \qquad x\to 0,\ n = 0, 1, 2, \cdots; \tag{4.5}$$

$$f^{(n)}(x) = o(x^{-n}), \qquad x\to \infty,\ n = 0, 1, 2, \cdots. \tag{4.6}$$

5. **The S_2 transform as an iterated Stieltjes transform.** The S_2 transform was obtained by formally changing the order of integration in

$$\int_{0+}^{\infty} \frac{dt}{x+t}\int_{0+}^{\infty} \frac{d\alpha(u)}{t+u}. \tag{5.1}$$

In this section we shall show that this formal process is not always permissible; that is, that (5.1) may converge when (1.3) does not. We shall show, however, that when (1.3) converges, (5.1) also converges, and we shall obtain necessary and sufficient conditions for the convergence of (5.1) to imply that of (1.3).

DEFINITION 5.1. *Let* $\alpha(t)$ *be a normalized function, of bounded variation on every* (ϵ, R), $(0<\epsilon<R<\infty)$. *Then the iterated integral* (5.1), *if it exists, is called an iterated Stieltjes transform*; $\alpha(t)$ *is called its determining function.*

LEMMA 5.1. *The function*

$$H(u) = \frac{1}{\log u}\log\frac{u}{u+\epsilon}$$

decreases if $0<\epsilon<e^{-1}$ *and* $0<u<e^{-1}-\epsilon$.

* G. H. Hardy and J. E. Littlewood, *Notes on the theory of series* (XI): *On Tauberian theorems*, Proceedings of the London Mathematical Society, (2), vol. 30 (1930), pp. 23–37; D. V. Widder, op. cit.

For the proof, we have

$$H'(u) = \frac{1}{\log u}\left(\frac{1}{u} - \frac{1}{u+\epsilon}\right) - \frac{1}{u(\log u)^2}\log\frac{u}{u+\epsilon}$$
$$= \frac{-1}{u\log u}\left\{\frac{u}{u+\epsilon} - \frac{\log(u+\epsilon)}{\log u}\right\},$$

which has the sign of the expression in the braces. For $0<u<e^{-1}$, the function $u\log u^{-1}$ increases; and if $0<u+\epsilon<e^{-1}$, then

$$(u+\epsilon)\log\frac{1}{u+\epsilon} > u\log\frac{1}{u},$$
$$\frac{\log(u+\epsilon)}{\log u} > \frac{u}{u+\epsilon},$$

and we have $H'(u)<0$.

THEOREM 5.2. *If the S_2 transform*

$$\int_{0+}^{\infty}\frac{\log(x/t)}{x-t}\,d\alpha(t) \tag{5.2}$$

converges, then the iterated Stieltjes transform

$$\int_{0+}^{\infty}\frac{dt}{x+t}\int_{0+}^{\infty}\frac{d\alpha(u)}{t+u}$$

converges, and the two are equal.

By Lemma 2.5 and an application of Lemmas 2.3 and 2.4, it can be shown that

$$\int_{0+}^{\infty}\frac{d\alpha(u)}{t+u}$$

converges; by Theorem 4.1, it converges uniformly on (ϵ, R), $(0<\epsilon<R<\infty)$. Hence for $x>0$,

$$\int_{\epsilon}^{R}\frac{dt}{x+t}\int_{0+}^{\infty}\frac{d\alpha(u)}{t+u} = \int_{0+}^{\infty}d\alpha(u)\int_{\epsilon}^{R}\frac{dt}{(x+t)(u+t)}$$
$$= \int_{0+}^{\infty}\log\frac{x+R}{u+R}\,\frac{d\alpha(u)}{u-x} - \int_{0+}^{\infty}\log\frac{x+\epsilon}{u+\epsilon}\,\frac{d\alpha(u)}{u-x}$$
$$= I + J.$$

We shall show that $\lim_{R\to\infty} I=0$, $\lim_{\epsilon\to 0} J=f(x)$.

We take any $x>0$ and fix it throughout the discussion. Then

$$
\begin{aligned}
J - f(x) &= \int_{0+}^{\infty} \frac{1}{x-u} \log\left(\frac{x+\epsilon}{x}\,\frac{u}{u+\epsilon}\right) d\alpha(u) \\
&= \log \frac{x+\epsilon}{x} \int_{0+}^{x-\delta} \frac{d\alpha(u)}{x-u} - \left(\int_{0+}^{\zeta} + \int_{\zeta}^{x-\delta}\right) \log \frac{u+\epsilon}{u}\,\frac{d\alpha(u)}{x-u} \qquad (5.3)\\
&\quad + \left(\int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty}\right) \log\left(\frac{x+\epsilon}{x}\,\frac{u}{u+\epsilon}\right)\frac{d\alpha(u)}{x-u} \\
&= J_1 + J_2 + J_3 + J_4 + J_5, \qquad 0 < \zeta < x - \delta < x.
\end{aligned}
$$

In J_4

$$
\left|\frac{1}{x-u}\log\left(\frac{x+\epsilon}{x}\,\frac{u}{u+\epsilon}\right)\right| = \left|\frac{1}{x-u}\int_u^x\left(\frac{1}{t+\epsilon} - \frac{1}{t}\right)dt\right| \leqq \frac{\epsilon}{(x-\delta)(x-\delta+\epsilon)}.
$$

Fix δ, $(0<\delta<x)$; then

$$
|J_4| \leqq \frac{\epsilon}{(x-\delta)(x-\delta+\epsilon)} \int_{x-\delta}^{x+\delta} |d\alpha(u)| = o(1), \qquad \epsilon \to 0.
$$

With fixed δ, it is clear that $J_1 = o(1)$, $(\epsilon \to 0)$.

For $u>0$,

$$
\log\left(\frac{x+\epsilon}{x}\,\frac{u+\epsilon}{u}\right)
$$

is a positive, decreasing function of u; since

$$
\int_{x+\delta}^{\infty} \frac{d\alpha(u)}{x-u}
$$

converges, the integral J_5 converges, by Lemma 2.3, uniformly with respect to ϵ, $(0<\epsilon<1)$. Then we may let $\epsilon \to 0$ under the integral sign in J_5 and thus obtain $J_5 = o(1)$, $(\epsilon \to 0)$.

Since $\epsilon \to 0$, we may suppose that $\epsilon < (2e)^{-1}$. Let $\zeta = (2e)^{-1}$. Then $\zeta + \epsilon < 1/e$, and by Lemma 5.1, $H(u)$ decreases for $0<u<\zeta$. But $H(u) \geqq 0$, $H(u)$ is bounded (uniformly with respect to ϵ for $0<\epsilon<(2e)^{-1}$, $\lim_{\epsilon\to 0} H(u) = 0$, and

$$
J_2 = \int_{0+}^{\zeta} H(u)\,\frac{\log(1/u)}{x-u}\,d\alpha(u).
$$

By Lemma 2.4, this integral converges uniformly with respect to ϵ, since

$$\int_{0+}^{\zeta} \frac{\log (1/u)}{x-u}\, d\alpha(u)$$

converges, and we may therefore let $\epsilon \to 0$ under the integral sign. Hence $J_2 = o(1)$, $(\epsilon \to 0)$.

Finally, $\log [(u+\epsilon)/u]$ is positive decreasing in J_3; and

$$\begin{aligned} |J_3| &\leq \log \frac{\zeta+\epsilon}{\zeta} \operatorname*{u.b.}_{\zeta \leq \zeta' \leq x-\delta} \left| \int_{\zeta}^{\zeta'} \frac{d\alpha(u)}{x-u} \right| \\ &= o(1), \qquad \epsilon \to 0. \end{aligned}$$

To show that $I \to 0$, we set $x = y^{-1}$, $R = \eta^{-1}$, and $t = u^{-1}$. Then

$$I = \int_{0+}^{\infty} \log \left(\frac{\eta + y}{\eta + u} \frac{u}{y} \right) \frac{d\beta(u)}{y-u}, \qquad \beta(u) = -y \int_{0+}^{u} t\, d\alpha(t^{-1}),$$

which is an expression of the same form as (5.3); the S_2 transform

$$\int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, d\beta(t)$$

converges if (5.2) does, as the change of variable $u = t^{-1}$ shows. Hence $\lim_{R\to\infty} I = 0$.

THEOREM 5.3. *If the iterated Stieltjes transform*

$$\int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u} \tag{5.4}$$

converges, the S_2 transform

$$\int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, d\alpha(t)$$

converges (*and is equal to* (5.4)) *if and only if*

$$\alpha(t) - \alpha(0+) = o(-1/\log t), \qquad t \to 0, \tag{5.5}$$

$$\int_{t}^{\infty} u^{-1} d\alpha(u) = o(1/\log t), \qquad t \to \infty. \tag{5.6}$$

If the S_2 transform converges, it is equal to (5.4) by the previous theorem; conditions (5.5) and (5.6) are satisfied, by Theorem 3.1.

To establish the converse, we take any fixed $x > 0$ and consider separately

$$\int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{x} \frac{d\alpha(u)}{t+u}, \tag{5.7}$$

$$\int_{0+}^{\infty} \frac{dt}{x+t} \int_{x}^{\infty} \frac{d\alpha(u)}{t+u}. \tag{5.8}$$

For each of these integrals, we must show that the order of integration can be changed. We need to do this only for (5.8). For, in (5.7) let us set $t=s^{-1}$, $u=v^{-1}$, $x=y^{-1}$. Then (5.7) becomes

$$-y\int_{0+}^{\infty} \frac{ds}{y+s} \int_{y}^{\infty} \frac{v d\alpha(v^{-1})}{s+v}, \tag{5.9}$$

which has the same form as (5.8), if we write

$$\beta(u) = -y\int_{y}^{u} v d\alpha(v^{-1}), \qquad \beta(y) = 0;$$

also

$$\int_{t}^{\infty} u^{-1} d\beta(u) = y\alpha(t^{-1}) = o(1/\log t), \qquad t \to \infty,$$

if (5.5) is satisfied. Thus if we may change the order of integration in (5.8) when (5.6) is satisfied, then we may change the order in (5.9) if (5.5) is satisfied, and hence in (5.7).

We consider

$$I(R) = \int_{0+}^{R} \frac{dt}{x+t} \int_{x}^{\infty} \frac{d\alpha(u)}{t+u},$$

which by hypothesis approaches a limit as $R \to \infty$. The integral

$$\int_{x}^{\infty} \frac{d\alpha(u)}{t+u}$$

converges uniformly for $0 \leqq t \leqq R$, as one sees by applying Theorem 4.1, after setting $u=v+x$, $t=s-x$. Therefore

$$\begin{aligned} I(R) &= \int_{x}^{\infty} d\alpha(u) \int_{0}^{R} \frac{dt}{(x+t)(u+t)} \\ &= \int_{x}^{\infty} \left\{ \log \frac{x+R}{u+R} + \log \frac{u}{x} \right\} \frac{d\alpha(u)}{u-x}. \end{aligned}$$

We wish to show that

$$J(R) = \int_{x}^{R} \frac{\log (u/x)}{u-x} d\alpha(u)$$

has the same limit when $R \to \infty$ as $I(R)$. We consider, for $R > 2x$, the difference

$$I(R) - J(R) = \log \frac{x+R}{x} \int_R^\infty \frac{d\alpha(u)}{u-x} - \int_R^\infty \log \frac{u+R}{u} \frac{d\alpha(u)}{u-x}$$
$$+ \left(\int_x^{2x} + \int_{2x}^R \right) \log \frac{x+R}{u+R} \frac{d\alpha(u)}{u-x}$$
$$= I_1 + I_2 + I_3 + I_4.$$

We note that the integral I_1 converges by (5.6) combined with Lemma 2.3. Using Lemma 2.2 and applying (5.6), we obtain

$$| I_1 | = O(\log R) \left| \int_R^\infty \frac{u}{u-x} \frac{d\alpha(u)}{u} \right|$$
$$\leqq O(\log R) \frac{R}{R-x} \operatorname*{u.b.}_{R' \geqq R} \left| \int_R^{R'} \frac{d\alpha(u)}{u} \right|$$
$$= o(1), \qquad R \to \infty.$$

It is a simple consequence of (5.6) that

$$\beta(t) \equiv \int_t^\infty \frac{d\alpha(u)}{u-x} = o(1/\log t), \qquad t \to \infty. \tag{5.10}$$

Therefore, since $\log [(u+R)/u]$ is positive decreasing, $(u>R)$, we have

$$| I_2 | \leqq \log 2 \operatorname*{u.b.}_{R' \geqq R} \left| \int_R^{R'} \frac{d\alpha(u)}{u-x} \right| = o(1), \qquad R \to \infty.$$

Also,

$$I_4 = \int_{2x}^R \log \frac{x+R}{u+R} d\beta(u)$$
$$= \beta(R) \log \frac{x+R}{2R} - \beta(2x) \log \frac{x+R}{2x+R} + \int_{2x}^R \frac{\beta(u)}{u+R} du$$
$$= o(1), \qquad R \to \infty,$$

by use of (5.10).

Finally,

$$| I_3 | = \left| \int_x^{2x} \frac{\log (x+R) - \log (u+R)}{x-u} d\alpha(u) \right|$$
$$\leqq \frac{1}{x+R} \int_x^{2x} | d\alpha(u) | = o(1), \qquad R \to \infty.$$

To complete our discussion of the relations between the S_2 transform and the iterated Stieltjes transform, we need to establish the following theorem:

THEOREM 5.4. *There exists a function $\alpha(t)$ of bounded variation on $(0, \infty)$, such that*

$$\int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u} \tag{5.11}$$

converges, and

$$\int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, d\alpha(t) \tag{5.12}$$

diverges.

Let $\{u_n\}$, $\{u_n'\}$, $(n=1, 2, \cdots)$, be sequences of points such that $0<u_{n+1}'<u_n<u_n'<1$, and such that the series

$$\sum_{n=1}^{\infty} \frac{1}{\log u_n}, \qquad \sum_{n=1}^{\infty} \frac{\log u_n' - \log u_n}{\log u_n}$$

converge (for example, $u_n=2^{-n^2}$, $u_n'=2^{-n^2+n}$). Then the function

$$\alpha(u) = \begin{cases} -(\log u_n)^{-1}, & u_n < u < u_n', \\ -(2\log u_n)^{-1}, & u = u_n, \qquad u = u_n', \\ 0, & \text{elsewhere}, \end{cases}$$

has the desired properties.

The total variation of $\alpha(u)$ on $(0, \infty)$ is

$$\sum_{n=1}^{\infty} \frac{-2}{\log u_n} < \infty.$$

Moreover,

$$\int_{0+}^{1} \frac{\alpha(u)}{u}\, du = \sum_{n=1}^{\infty} \frac{-1}{\log u_n} \int_{u_n}^{u_n'} \frac{du}{u} = \sum_{n=1}^{\infty} \frac{\log u_n - \log u_n'}{\log u_n} < \infty.$$

Also, $\alpha(u) \geqq 0$; $\alpha(0)=\alpha(0+)=\alpha(1)=0$; $\alpha(u)-\alpha(0+)\neq o(-1/\log u)$, $(u\to 0)$, since $\alpha(u_n)=(-2\log u_n)^{-1}$ (but $\alpha(u)=O(-1/\log u)$, $(u\to 0)$), and

$$\begin{aligned} \int_{0+}^{1} \frac{\alpha(u)}{u}\, du &= \int_{0+}^{1} \alpha(u) du \int_{0}^{\infty} \frac{dt}{(u+t)^2} \\ &= \int_{0}^{\infty} dt \int_{0+}^{1} \frac{\alpha(u) du}{(u+t)^2} \\ &= \int_{0}^{\infty} dt \int_{0+}^{1} \frac{d\alpha(u)}{u+t} \end{aligned}$$

(the change of order of integration is legitimate because the integrand is non-negative). Hence, by use of Lemmas 2.3 and 2.4, (5.11) converges. But (5.12) must diverge, since (3.1) is not satisfied.

Chapter II. Inversion of the transforms

6. The inversion operator. In the remainder of this paper, unless the contrary is specified, all quantities are to be real, and the domain of all functions is $(0, \infty)$.

Definition 6.1.* *For a function $f(x)$ of class C^{2k-1}, an operator $L_{k,x}[f(x)]$ is defined by*

$$L_{k,x}[f(x)] = c_k(-x)^{k-1}[x^k f(x)]^{(2k-1)}, \qquad k = 1, 2, \cdots, \tag{6.1}$$

$$c_1 = 1, \qquad c_k = \frac{1}{k!(k-2)!}, \qquad k = 2, 3, \cdots. \tag{6.2}$$

Definition 6.2. *For a function $f(x)$ of class C^{4k-2}, an operator $H_{k,x}[f(x)]$ is defined by*

$$H_{k,x}[f(x)] = L_{k,x}\{L_{k,x}[f(x)]\}. \tag{6.3}$$

Theorem 6.1. *If $f(x)$ is an iterated Stieltjes transform*

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u}, \tag{6.4}$$

then

$$H_{k,x}[f(x)] = \int_{0+}^{\infty} F_k(u, x)\, d\alpha(u), \qquad k = 2, 3, \cdots, \tag{6.5}$$

where

$$F_k(u, x) = d_k^2 x^{k-1} u^k \int_0^{\infty} \frac{t^{2k-1}dt}{(x+t)^{2k}(u+t)^{2k}}, \tag{6.6}$$

$$d_k = (2k-1)!\, c_k. \tag{6.7}$$

We have

$$f(x) = \int_{0+}^{\infty} \frac{\phi(t)dt}{x+t}, \qquad \phi(t) = \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u};$$

by Theorem 4.1, $f(x)$ and $\phi(t)$ are of class C^∞; their derivatives may be evaluated by Leibniz' rule; and, for $n = 0, 1, 2, \cdots$,

$$\begin{aligned} \phi^{(n)}(t) &= o(t^{-n-1}), & t &\to 0, \\ \phi^{(n)}(t) &= o(t^{-n}), & t &\to \infty. \end{aligned} \tag{6.8}$$

---

* D. V. Widder, op. cit.

Now for $k \geqq 2$

$$\frac{1}{k!}[x^k f(x)]^{(k)} = \frac{1}{k!}\int_{0+}^{\infty} \frac{\partial^k}{\partial x^k}\left(\frac{x^k}{x+t}\right)\phi(t)dt$$
$$= \int_0^{\infty} \frac{t^k \phi(t)}{(x+t)^{k+1}} dt.$$

If we integrate by parts k times, we find, by (6.8), that the integrated terms all vanish, so that

$$[x^k f(x)]^{(k)} = \int_0^{\infty} \frac{[t^k\phi(t)]^{(k)}}{x+t} dt.$$

Thus

$$x^{k-1}[x^k f(x)]^{(2k-1)} = (-1)^{k-1}(k-1)!\int_0^{\infty} \frac{x^{k-1}[t^k\phi(t)]^{(k)}}{(x+t)^k} dt$$
$$= (-1)^{k-1}\int_0^{\infty} [t^k\phi(t)]^{(k)} \frac{\partial^{k-1}}{\partial t^{k-1}}\left(\frac{t^{k-1}}{x+t}\right) dt.$$

We integrate by parts $k-1$ times; the integrated terms all vanish, and we obtain

$$L_{k,x}[f(x)] = \int_0^{\infty} \frac{L_{k,t}[\phi(t)]}{x+t} dt. \tag{6.9}$$

If

$$g(x) = \int_{0+}^{\infty} \frac{d\psi(t)}{x+t},$$

then

$$L_{k,x}[g(x)] = d_k x^{k-1}\int_{0+}^{\infty} \frac{t^k d\psi(t)}{(x+t)^{2k}}.$$

Applying this formula twice to (6.9), we obtain

$$\begin{aligned} H_{k,x}[f(x)] &= d_k x^{k-1}\int_0^{\infty} \frac{t^k L_{k,t}[\phi(t)]dt}{(x+t)^{2k}} \\ &= d_k^2\, x^{k-1}\int_0^{\infty} \frac{t^{2k-1}dt}{(x+t)^{2k}}\int_{0+}^{\infty} \frac{u^k d\alpha(u)}{(u+t)^{2k}}. \end{aligned} \tag{6.10}$$

We now wish to change the order of integration in (6.10). Writing

$$A(u, t) = \frac{u^k}{(u+t)^{2k}}$$

we see, by use of (4.3) and (4.4), that

$$\int_{0+}^{\infty} [\alpha(u) - \alpha(0+)] \frac{\partial}{\partial u} A(u, t) du$$

exists, and that

$$(6.11)\qquad \begin{aligned} -\int_{0+}^{\infty} [\alpha(u) - \alpha(0+)] \frac{\partial}{\partial u} A(u, t) du &= \int_{0+}^{\infty} A(u, t) d[\alpha(u) - \alpha(0+)] \\ &= \int_{0+}^{\infty} \frac{u^k d\alpha(u)}{(t+u)^{2k}}. \end{aligned}$$

Now consider the integral

$$d_k^2 x^{k-1} \int_{0+}^{\infty} u^k d\alpha(u) \int_0^{\infty} \frac{t^{2k-1} dt}{(x+t)^{2k}(u+t)^{2k}} = \int_{0+}^{\infty} F_k(u, x) d[\alpha(u) - \alpha(0+)],$$

$k \geqq 2$. For we anticipate the inequalities of Lemma 7.3; combined with (4.3) and (4.4), they show that the last integral exists and is equal to

$$\begin{aligned} &-\int_{0+}^{\infty} [\alpha(u) - \alpha(0+)] \frac{\partial}{\partial u} F_k(u, x) du \\ &\qquad = -k d_k^2 x^{k-1} \int_{0+}^{\infty} [\alpha(u) - \alpha(0+)] u^{k-1} du \int_0^{\infty} \frac{t^{2k-1}(t-u) dt}{(x+t)^{2k}(u+t)^{2k+1}}. \end{aligned}$$

The repeated integral is easily seen (compare (7.5), (7.6)) to be dominated by

$$k \int_0^{\infty} | \alpha(u) - \alpha(0+) | u^{-1} F_k(u, x) du,$$

which (because of (4.3), (4.4), and Lemma 7.3) converges ($k \geqq 2$); hence the order of integration in the repeated integral can be changed, so that

$$\int_{0+}^{\infty} F_k(u, x) d\alpha(u) = -d_k^2 x^{k-1} \int_0^{\infty} \frac{t^{2k-1} dt}{(x+t)^{2k}} \int_{0+}^{\infty} [\alpha(u) - \alpha(0+)] \frac{\partial}{\partial u} A(u, t) du.$$

Referring to (6.11), we obtain (6.5).

COROLLARY 6.1.1. *For* $k \geqq 2$,

$$H_{k,x}\left[\frac{\log (x/t)}{x-t}\right] = F_k(t, x).$$

We may write

$$\frac{\log (x/t)}{x-t} = \int_{0+}^{\infty} \frac{\log (x/u)}{x-u} d\alpha(u), \quad \alpha(u) = \begin{cases} 0, & 0 \leqq u < t, \\ 1/2, & u = t, \\ 1, & u > t. \end{cases}$$

Then by Theorem 5.2

$$\frac{\log (x/t)}{x-t} = \int_{0+}^{\infty} \frac{du}{x+u} \int_{0+}^{\infty} \frac{d\alpha(v)}{t+v};$$

by Theorem 6.1

$$H_{k,x}\left[\frac{\log (x/t)}{x-t}\right] = \int_{0+}^{\infty} F_k(u, x) d\alpha(u) = F_k(t, x), \qquad k \geqq 2.$$

7. **Properties of the function $F_k(u, x)$.** We have the following lemma:

LEMMA 7.1. *If $m=1, 2, \cdots; n=2, 3, \cdots; n>m$, then*

$$\int_0^{\infty} \frac{u^{m-1}du}{(t+u)^n} = \frac{(m-1)!(n-m-1)!}{t^{n-m}(n-1)!}.$$

This is the familiar formula for the beta function.

LEMMA 7.2. *If $k=2, 3, \cdots$, then*

$$\int_0^{\infty} F_k(u, x) du = 1,$$

$$x\int_0^{\infty} u^{-1}F_k(u, x) du = \int_0^{\infty} F_k(u, x) dx = \left(\frac{k-1}{k}\right)^2.$$

These formulas are obtained by applying Lemma 7.1 twice to each of the repeated integrals in question, after changing the order of integration.

LEMMA 7.3. *If $k=2, 3, \cdots$ and $x>0$ is fixed, then*

$$F_k(u, x) = O(u^{-k+1}), \qquad u \to \infty, \tag{7.1}$$

$$F_k(u, x) = O(u^{k-1}), \qquad u \to 0, \tag{7.2}$$

$$\frac{\partial}{\partial u} F_k(u, x) = O(u^{-k}), \qquad u \to \infty, \tag{7.3}$$

$$\frac{\partial}{\partial u} F_k(u, x) = O(u^{k-2}), \qquad u \to 0. \tag{7.4}$$

Since $u/(u+t)<1$, $1/(u+t)<t^{-1}$, and $t/(u+t)<1$, $(u>0, t>0)$, we have

$$u^{k-1}F_k(u, x) \leqq d_k^2 x^{k-1} \int_0^{\infty} \frac{t^{2k-2}dt}{(x+t)^{2k}},$$

which gives (7.1). We have also the following relation from which (7.2) follows:

$$F_k(u, x) \leqq d_k^2 x^{k-1}u^k \int_0^{\infty} \frac{dt}{(x+t)^{2k}(u+t)} \leqq d_k^2 x^{k-1}u^{k-1} \int_0^{\infty} \frac{dt}{(x+t)^{2k}}.$$

Since

$$\frac{\partial}{\partial u} F_k(u, x) = k d_k^2 u^{k-1} x^{k-1} \int_0^\infty \frac{t^{2k-1}(t-u)dt}{(x+t)^{2k}(u+t)^{2k+1}}, \tag{7.5}$$

and since for $u>0$ and $t>0$ the inequality $|t-u|/(u+t)<1$ holds, it follows that

$$\left| \frac{\partial}{\partial u} F_k(u, x) \right| \leq \frac{k}{u} F_k(u, x); \tag{7.6}$$

hence (7.3) and (7.4) follow from (7.6) combined with (7.1) and (7.2).

LEMMA 7.4. *$F_k(u, x)$, as a function of u, increases for $u<x$ and decreases for $u>x$.*

It is sufficient to establish this for $x=1$, since $F_k(u, x)$ is homogeneous of degree -1, so that

$$F_k(u, x) = x^{-1} F_k(x^{-1}u, 1).$$

We have, from (7.5),

$$\begin{aligned} \frac{1}{kd_k^2} \frac{\partial}{\partial u} F_k(u, 1) &= u^{k-1} \int_0^\infty \frac{t^{2k}dt}{(1+t)^{2k}(u+t)^{2k+1}} - u^k \int_0^\infty \frac{t^{2k-1}dt}{(1+t)^{2k}(u+t)^{2k+1}} \\ &= I_1 - I_2. \end{aligned}$$

If we make the change of variable $t=u/s$, and replace s by t in the result, we find that

$$\begin{aligned} I_1 &= u^{k-1} \int_0^\infty \frac{t^{2k-1}dt}{(1+t)^{2k+1}(u+t)^{2k}}, \\ I_1 - I_2 &= u^{k-1} \int_0^\infty \frac{t^{2k-1}}{(1+t)^{2k}(u+t)^{2k}} \left[\frac{1}{1+t} - \frac{u}{u+t} \right] dt \\ &= (1-u)u^{k-1} \int_0^\infty \frac{t^{2k}dt}{(1+t)^{2k+1}(u+t)^{2k+1}}, \end{aligned}$$

which has the sign of $(1-u)$.

8. **Some preliminary limits.** We establish the following lemma:

LEMMA 8.1. *If $0<y<1$, then*

$$\lim_{k\to\infty} kd_k \int_0^y \frac{u^{k-1}du}{(u+1)^{2k}} = 0. \tag{8.1}$$

If $0<y<1$, then the function $u(u+1)^{-2}$, which has a single maximum (at $u=1$), increases on $(0, y)$, and

$$0 \leqq kd_k \int_0^y \frac{u^{k-1}du}{(u+1)^{2k}} < kd_k \frac{y^{k-1}}{(y+1)^{2k-2}} \int_0^y \frac{du}{(u+1)^2};$$

the last expression approaches zero ($k \to \infty$), since it is the general term of a convergent infinite series. In fact, the test ratio for the series is

$$\frac{2(2k+1)}{k-1} \frac{y}{(y+1)^2}$$

which approaches a limit less than unity ($k \to \infty$).

LEMMA 8.2. *Let*

$$\begin{aligned} H_k(y) &= \int_0^y F_k(1, x)dx \\ &= d_k^2 \int_0^y x^{k-1}dx \int_0^\infty \frac{t^{2k-1}dt}{(x+t)^{2k}(1+t)^{2k}}. \end{aligned} \tag{8.2}$$

Then

$$\lim_{k\to\infty} kH_k(y) = 0, \qquad 0 \leqq y < 1, \tag{8.3}$$

$$\lim_{k\to\infty} k[1 - H_k(y)] = 0, \qquad y > 1, \tag{8.4}$$

$$\lim_{k\to\infty} H_k(1) = \tfrac{1}{2}. \tag{8.5}$$

We consider first $0<y<1$. In (8.2) change the order of integration and make the change of variable $x=ut$. Then

$$d_k^{-2}H_k(y) = \int_0^\infty dt \int_0^{y/t} \frac{(ut)^{k-1}du}{(u+1)^{2k}(t+1)^{2k}} \tag{8.6}$$

$$\leqq \left(\int_0^\infty \int_0^\infty - \int_{y^{1/2}}^\infty \int_{y^{1/2}}^\infty \right) \frac{u^{k-1}t^{k-1}dudt}{(u+1)^{2k}(t+1)^{2k}}, \tag{8.7}$$

since the integrand is nonnegative, and the domain of integration in (8.7) includes that in (8.6). Thus

$$\begin{aligned} d_k^{-2}H_k(y) &\leqq \left\{ \int_0^\infty \frac{t^{k-1}dt}{(t+1)^{2k}} - \int_{y^{1/2}}^\infty \frac{t^{k-1}dt}{(t+1)^{2k}} \right\} \\ &\quad \cdot \left\{ \int_0^\infty \frac{t^{k-1}dt}{(t+1)^{2k}} + \int_{y^{1/2}}^\infty \frac{t^{k-1}dt}{(t+1)^{2k}} \right\} \\ &\leqq 2 \int_0^{y^{1/2}} \frac{t^{k-1}dt}{(t+1)^{2k}} \int_0^\infty \frac{u^{k-1}du}{(u+1)^{2k}}; \end{aligned}$$

and

$$H_k(y) \leqq 2d_k \int_0^{y^{1/2}} \frac{u^{k-1}dt}{(t+1)^{2k}},$$

since by Lemma 7.1 it is seen that

$$d_k \int_0^\infty \frac{u^{k-1}du}{(u+1)^{2k}} = \frac{k-1}{k} < 1.$$

Then by Lemma 8.1, $\lim_{k\to\infty} kH_k(y) = 0$, $(0<y<1)$.

Now consider $y>1$. We have by Lemma 7.2

$$\int_0^\infty F_k(1, x)dx = \left(\frac{k-1}{k}\right)^2,$$

$$\left(\frac{k-1}{k}\right)^2 - H_k(y) = \int_y^\infty F_k(1, x)dx = \int_0^{y^{-1}} x^{-2}F_k(1, x^{-1})dx.$$

But

$$F_k(1, x) = x^{-1}F_k(x^{-1}, 1) = x^{-2}F_k(1, x^{-1}), \tag{8.8}$$

and (8.4) thus follows by what has already been established.

Finally consider $H_k(1)$. Using (8.8), we obtain

$$H_k(1) = \int_0^1 F_k(1, x)dx = \int_1^\infty x^{-2}F_k(1, x^{-1})dx = \int_1^\infty F_k(1, x)dx;$$

$$2H_k(1) = \int_0^\infty F_k(1, x)dx = \left(\frac{k-1}{k}\right)^2 \to 1, \qquad k \to \infty.$$

LEMMA 8.3. *If $u \neq x$, then*

$$\lim_{k\to\infty} F_k(u, x) = 0. \tag{8.9}$$

It is sufficient to consider $F_k(u, 1)$, because $F_k(u, x) = x^{-1}F_k(u/x, 1)$. Since $F_k(0, 1) = 0$, we have by (7.6)

$$0 \leqq F_k(y, 1) = \int_0^y \frac{\partial}{\partial u} F_k(u, 1)du \leqq k \int_0^y u^{-1}F_k(u, 1)du.$$

That is,

$$0 \leqq F_k(y, 1) \leqq k \int_0^y F_k(1, u)du = kH_k(y) = o(1), \qquad k \to \infty,$$

if $0<y<1$, by Lemma 8.2. Since

$$F_k(x, u) = xu^{-1}F_k(u, x) = u^{-1}F_k(xu^{-1}, 1),$$

we have $F_k(y^{-1}, 1) = F_k(y, 1)$; and (8.9) for $u/x<1$ implies (8.9) for $u/x>1$.

It is interesting to compare Lemma 8.3 with the following lemma:

LEMMA 8.4. *If $x>0$, then*

$$F_k(x, x) \sim \frac{1}{x}\left(\frac{k}{8\pi}\right)^{1/2}, \qquad k \to \infty.$$

For,

$$F_k(x, x) = d_k^2 x^{2k-1} \int_0^\infty \frac{t^{2k-1}dt}{(x+t)^{4k}} = \frac{1}{x} d_k^2 \frac{(2k-1)!(2k-1)!}{(4k-1)!},$$

and an application of Stirling's formula gives the result.

9. **A singular integral.** We prove the following lemma:

LEMMA 9.1. *If $u^m\beta(u)$ is bounded and integrable on $0 \leqq u \leqq x$ for some integer $m \geqq 0$, then for $0<\delta<x$*

$$I_k = \int_0^{x-\delta} \beta(u) \frac{\partial}{\partial u} F_k(u, x)du \to 0, \qquad k \to \infty.$$

That the integrand is integrable for sufficiently large k follows from Lemma 7.3. By (7.6)

$$|I_k| \leqq k \int_0^{x-\delta} u^{-1} |\beta(u)| F_k(u, x)du.$$

Set $u = xv$, and assume $|u^m\beta(u)| < B$, $(0 \leqq u \leqq x)$. For $k>m+1$, we have

$$\begin{aligned}
|I_k| &\leqq k \int_0^{1-x^{-1}\delta} v^{-1} |\beta(xv)| F_k(vx, x)dv \\
&= kx^{-1} \int_0^{1-x^{-1}\delta} v^{-1} |\beta(xv)| F_k(v, 1)dv \\
&\leqq x^{-m-1}Bk \int_0^{1-x^{-1}\delta} v^{-m-1}F_k(v, 1)dv \\
&= x^{-m-1}Bkd_k^2 \int_0^{1-x^{-1}\delta} v^{k-m-1}dv \int_0^\infty \frac{t^{2k-1}dt}{(1+t)^{2k}(t+v)^{2k}} \\
&\leqq x^{-m-1}Bkd_k^2 \int_0^{1-x^{-1}\delta} v^{k-m-1}dv \int_0^\infty \frac{t^{2k-2m-1}dt}{(1+t)^{2k-2m}(t+v)^{2k-2m}} \\
&= x^{-m-1}Bkd_k^2 d_{k-m}^{-2} H_{k-m}(1 - x^{-1}\delta) \\
&= o(1), \qquad k \to \infty,
\end{aligned}$$

by Lemma 8.2, since $d_k/d_{k-m} = O(1)$, $(k \to \infty)$.

LEMMA 9.2. *If $\beta(u)$ is bounded and integrable on (x, R) for every $R>x>0$, and if $\beta(u)=O(u^m)$, $(u\to\infty)$, for some integer $m\geqq 0$, then for every $\delta>0$,*

$$J_k = \int_{x+\delta}^{\infty} \beta(u) \frac{\partial}{\partial u} F_k(u, x) du \to 0, \qquad k\to\infty.$$

We proceed as in Lemma 9.1. Assume $|\beta(u)u^{-m}|<B$ for $u\geqq x$. For $k>m+1$, we then have

$$\begin{aligned}
|J_k| &\leqq k\int_{x+\delta}^{\infty} u^{-1}|\beta(u)|F_k(u, x)du \\
&\leqq kBx^{m-1}\int_{1+x^{-1}\delta}^{\infty} v^{m-1}F_k(v, 1)dv \\
&= kBx^{m-1}d_k^2\int_{1+x^{-1}\delta}^{\infty} v^{k+m-1}dv\int_0^{\infty}\frac{t^{2k-1}dt}{(1+t)^{2k}(t+v)^{2k}} \\
&\leqq kBx^{m-1}d_k^2\int_{1+x^{-1}\delta}^{\infty} v^{k-m-1}dv\int_0^{\infty}\frac{t^{2k-2m-1}dt}{(1+t)^{2k-2m}(t+v)^{2k-2m}} \\
&= kBx^{m-1}d_k^2 d_{k-m}^{-2}\left\{\left(\frac{k-m-1}{k-m}\right)^2 - H_{k-m}(1+x^{-1}\delta)\right\} \\
&= o(1), \qquad k\to\infty.
\end{aligned}$$

THEOREM 9.3. *If $\phi(u)$ is integrable on every (ϵ, R), $(0<\epsilon<R<\infty)$, and if there exist integers $m\geqq 0$, $n\geqq 0$, such that*

$$\gamma(u) \equiv \int_1^u \phi(t)dt = \begin{cases} O(u^{-m}), & u\to 0, \\ O(u^n), & u\to\infty; \end{cases} \tag{9.1}$$

then for k sufficiently large, the integral

$$G_k(x) = \int_{0+}^{\infty} F_k(u, x)\phi(u)du \tag{9.2}$$

exists, and

$$\lim_{k\to\infty} G_k(x) = \phi(x) \tag{9.3}$$

for every $x>0$ for which either

$$\int_x^t |\phi(u)-\phi(x)|du = o(|t-x|), \qquad t\to x, \tag{9.4}$$

or $\phi(x+)$ and $\phi(x-)$ exist with

$$\phi(x) = \tfrac{1}{2}[\phi(x+)+\phi(x-)]. \tag{9.5}$$

We note that (9.4) is satisfied for almost all x, and, in particular, wherever $\phi(x)$ is continuous.*

We show first that (9.2) exists. For $R>1$,

$$\int_1^R F_k(u, x)\phi(u)du = \int_1^R F_k(u, x)d\gamma(u)$$
$$= F_k(R, x)\gamma(R) - \int_1^R \gamma(u)\frac{\partial}{\partial u}F_k(u, x)du.$$

By Lemma 7.3 and (9.1), if $k \geqq n+2$, this expression is

$$O(R^{-n-1})O(R^n) - \int_1^R O(u^n)O(u^{-n-2})du,$$

which approaches a limit as $R\to\infty$. A similar argument shows that

$$\int_{0+}^1 F_k(u, x)\phi(u)du, \qquad k \geqq m + 2,$$

converges.

Since by Lemma 7.2 $\int_0^\infty F_k(u, x)du = 1$, $(k \geqq 2)$, we have

$$D_k(x) \equiv G_k(x) - \phi(x) = \int_{0+}^\infty [\phi(u) - \phi(x)]F_k(u, x)du. \tag{9.6}$$

Let x be a point where (9.4) is satisfied, and set

$$\beta(t, x) = \int_x^t [\phi(u) - \phi(x)]du.$$

Then

$$D_k(x) = \int_{0+}^\infty F_k(u, x)d_u\beta(u, x) = -\int_0^\infty \beta(u, x)\frac{\partial}{\partial u}F_k(u, x)du,$$

the integrated terms vanishing (for k sufficiently large) by (9.1) and Lemma 7.3. Assuming (9.4), we find that $\beta(u, x) = o(|u-x|)$, $(u\to x)$; we can therefore choose δ, $(0<\delta<x)$, so that, $\epsilon>0$ being given,

$$|\beta(u, x)| \leqq \epsilon|x - u|, \qquad |x - u| \leqq \delta; \tag{9.7}$$

then

$$-D_k(x) = \left(\int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^\infty\right)\beta(u, x)\frac{\partial}{\partial u}F_k(u, x)du$$
$$= I_1 + I_2 + I_3.$$

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* See, for example, E. C. Titchmarsh, *The Theory of Functions*, 1932, p. 364.

By using (9.1), we see that $\beta(u, x)$, as a function of u, satisfies the hypotheses of Lemmas 9.1 and 9.2, and hence that

$$\lim_{k\to\infty} (I_1 + I_3) = 0.$$

Since, by Lemma 7.4, $F_k(u, x)$ increases for $u<x$ and decreases for $u>x$,

$$|I_2| \leqq \int_{x-\delta}^{x} |\beta(u, x)|\, d_uF_k(u, x) - \int_{x}^{x+\delta} |\beta(u, x)|\, d_uF_k(u, x);$$

and if we use (9.7),

$$\begin{aligned} |I_2| &\leqq \epsilon\int_{x-\delta}^{x} (x-u)d_uF_k(u, x) - \epsilon\int_{x}^{x+\delta} (u-x)d_uF_k(u, x) \\ &= -\epsilon\delta F_k(x-\delta, x) - \epsilon\delta F_k(x+\delta, x) + \epsilon\int_{x-\delta}^{x+\delta} F_k(u, x)du \\ &< \epsilon\int_0^\infty F_k(u, x)du = \epsilon, \end{aligned}$$

since $F_k(u, x) \geqq 0$. Therefore

$$\limsup_{k\to\infty} |D_k(x)| \leqq \epsilon;$$

and since ϵ was arbitrary,

$$\lim_{k\to\infty} G_k(x) = \phi(x).$$

Now suppose that (9.5) is satisfied. Set

$$\theta(u) = \begin{cases} \phi(x-), & u < x, \\ \phi(x), & u = x, \\ \phi(x+), & u > x, \end{cases}$$

$$\omega(u) = \phi(u) - \theta(u);$$

then $\omega(u)$ is continuous at $u=x$, and $\omega(x)=0$. Hence

$$G_k(x) = \int_{0+}^{\infty} F_k(u, x)\omega(u)du + \int_0^\infty F_k(u, x)\theta(u)du;$$

$\omega(u)$ satisfies the hypotheses of the theorem and satisfies (9.4) at $u=x$. By what has already been established,

$$\lim_{k\to\infty} \int_{0+}^{\infty} F_k(u, x)\omega(u)du = \omega(x) = 0.$$

On the other hand,

$$\begin{aligned}\int_0^\infty F_k(u, x)\theta(u)du &= \phi(x-)\int_0^x F_k(u, x)du + \phi(x+)\int_x^\infty F_k(u, x)du\\ &= x\phi(x-)\int_0^1 F_k(ux, x)du + x\phi(x+)\int_1^\infty F_k(ux, x)du\\ &= \phi(x-)\int_0^1 F_k(u, 1)du + \phi(x+)\int_1^\infty F_k(u, 1)du;\end{aligned}$$

and

$$\int_0^\infty F_k(u, x)\theta(u)du = \phi(x+) + [\phi(x-) - \phi(x+)]\int_0^1 F_k(u, 1)du. \tag{9.8}$$

But

$$\begin{aligned}\int_0^1 F_k(u, 1)du &= \int_0^1 uF_k(1, u)du\\ &= H_k(1) - \int_0^1 (1-u)F_k(1, u)du;\end{aligned}$$

and for $0<\epsilon<1$,

$$\begin{aligned}\int_0^1 (1-u)F_k(1, u)du &= \int_0^{1-\epsilon} (1-u)F_k(1, u)du + \int_{1-\epsilon}^1 (1-u)F_k(1, u)du\\ &\leqq \int_0^{1-\epsilon} F_k(1, u)du + \epsilon\int_0^\infty F_k(1, u)du\\ &= H_k(1-\epsilon) + \epsilon\left(\frac{k-1}{k}\right)^2.\end{aligned}$$

By use of Lemma 8.2, we see that

$$\limsup_{k\to\infty}\int_0^1 (1-u)F_k(1, u)du \leqq \epsilon,$$

and hence that

$$\lim_{k\to\infty}\int_0^1 F_k(u, 1)du = \lim_{k\to\infty} H_k(1) = 1/2.$$

Thus (9.8) gives

$$\lim_{k\to\infty}\int_0^\infty F_k(u, x)\theta(u)du = \tfrac{1}{2}[\phi(x+) + \phi(x-)],$$

and the proof of Theorem 9.3 is complete.

COROLLARY 9.3.1. *The hypothesis* (9.1) *may be replaced by the requirement that the integrals*

$$\int_{0+}^{1} u^m \phi(u)du, \qquad \int_{1}^{\infty} u^{-n}\phi(u)du$$

exist, for some integers $m, n \geqq 0$.

We have only to verify (9.1). For $u > 1$, set

$$\psi(u) = \int_1^u t^{-n}\phi(t)dt.$$

Then

$$\gamma(u) = \int_1^u \phi(t)dt = \int_1^u t^n d\psi(t)$$

$$= \psi(u)u^n - n\int_1^u t^{n-1}\psi(t)dt = O(u^n), \qquad u \to \infty.$$

A similar argument applies when $u \to 0$.

10. **Inversion formulas.** We consider first the integral

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{\phi(u)du}{t+u}, \tag{10.1}$$

where $\phi(u)$ is integrable on every (ϵ, R), $(0 < \epsilon < R < \infty)$.

THEOREM 10.1. *If $f(x)$ has the form* (10.1), *then*

$$\phi(x) = \lim_{k\to\infty} H_{k,x}[f(x)] \tag{10.2}$$

for almost all $x > 0$.

Because of Theorem 5.2, this inversion formula serves also for S_2 transforms of the form

$$f(x) = \int_{0+}^{\infty} \frac{\log(x/t)}{x-t}\phi(t)dt.$$

By (4.3) the integral

$$\int_{0+}^{u} \phi(t)dt, \qquad u > 0,$$

exists; by (4.4)

$$\int_{0+}^{u} \phi(t)dt = o(u), \qquad u \to \infty.$$

Hence $\phi(t)$ satisfies the hypotheses of Corollary 9.3.1, and

$$\lim_{k\to\infty}\int_{0+}^{\infty} F_k(u, x)\phi(u)du = \phi(x)$$

for almost all x. But by Theorem 6.1

$$H_{k,x}[f(x)] = \int_{0+}^{\infty} F_k(u, x)\phi(u)du, \qquad k \geqq 2.$$

The same reasoning leads to the following corollary:

Corollary 10.1.1. *If $f(x)$ has the form* (10.1), *then* (10.2) *is true whenever* $\phi(x) = [\phi(x+)+\phi(x-)]/2$.

Theorem 10.2. *If $f(x)$ is an iterated Stieltjes transform of the form*

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t}\int_{0+}^{\infty}\frac{d\alpha(u)}{t+u}, \tag{10.3}$$

with $\alpha(t)$ a normalized function, of bounded variation on every (ϵ, R), $(0<\epsilon<R<\infty)$, *then*

$$\alpha(x) - \alpha(0+) = \lim_{k\to\infty}\int_{0+}^{x} H_{k,t}[f(x)]dt, \qquad x > 0. \tag{10.4}$$

Because of Theorem 5.2, this inversion formula serves also for the S_2 transform

$$f(x) = \int_{0+}^{\infty}\frac{\log (x/t)}{x-t}\, d\alpha(t).$$

We begin by showing that the integral in (10.4) is defined. We write $\beta(u) = \alpha(u) - \alpha(0+)$, $(u \geqq 0)$, and consider, for $y>\epsilon>0$ and $k\geqq 2$, the integral

$$\begin{aligned}\int_{\epsilon}^{y} H_{k,x}[f(x)]dx &= \int_{\epsilon}^{y} dx\int_{0+}^{\infty} F_k(u, x)d\beta(u)\\ &= -\int_{\epsilon}^{y} dx\int_{0}^{\infty}\beta(u)F_k(u, x)du;\end{aligned} \tag{10.5}$$

the integrated terms in the integration by parts vanish because of (4.3), (4.4), and Lemma 7.3. We may change the order of integration in equation (10.5) and obtain

$$\int_{\epsilon}^{y} H_{k,x}[f(x)]dx = -\int_{0}^{\infty}\beta(u)du\int_{\epsilon}^{y}\frac{\partial}{\partial u}F_k(u, x)dx \tag{10.6}$$

if

$$\int_\epsilon^y dx \int_0^\infty \left| \beta(u) \frac{\partial}{\partial u} F_k(u, x) \right| du \tag{10.7}$$

is finite. By (4.3), (4.4), there is a constant A such that

$$|\beta(u)| \leqq A(u+1), \qquad 0 \leqq u < \infty;$$

then, by use of (7.6) and Lemma 7.2,

$$\int_0^\infty \left| \beta(u) \frac{\partial}{\partial u} F_k(u, x) \right| du \leqq kA \int_0^\infty \frac{u+1}{u} F_k(u, x) du$$
$$= kA \left\{ 1 + \frac{1}{x} \left(\frac{k-1}{k} \right)^2 \right\}.$$

Hence the integral (10,7) is finite, and (10.6) is true. But $xF_k(u, x)$ is homogeneous of order zero, so that by Euler's theorem

$$\frac{\partial}{\partial u} F_k(u, x) = -\frac{1}{u} \frac{\partial}{\partial x} [xF_k(u, x)],$$

and (10.6) becomes

$$\int_\epsilon^y H_{k,x}[f(x)] dx = \int_0^\infty u^{-1} \beta(u) [yF_k(u, y) - \epsilon F_k(u, \epsilon)] du. \tag{10.8}$$

Write

$$I(\epsilon) = \epsilon \int_0^\infty u^{-1} \beta(u) F_k(u, \epsilon) du.$$

For $k \geqq 3$, at least, $I(\epsilon)$ is defined; we shall show that $\lim_{\epsilon \to 0} I(\epsilon) = 0$. We can find, given $\delta > 0$, constants η and $A > 0$ such that

$$|\beta(u)| < \tfrac{1}{2}\delta, \qquad 0 \leqq u \leqq \eta,$$
$$|\beta(u)| < Au, \qquad u \geqq \eta.$$

Then

$$I(\epsilon) \leqq \tfrac{1}{2}\epsilon\delta \int_0^\infty u^{-1} F_k(u, \epsilon) du + A\epsilon \int_0^\infty F_k(u, \epsilon) du = \frac{\delta}{2} \left(\frac{k-1}{k} \right)^2 + A\epsilon.$$

For $\epsilon < \delta/(2A)$, we have $|I(\epsilon)| < \delta$; and δ was arbitrary. Hence we may let $\epsilon \to 0$ in (10.8) and obtain

$$\int_{0+}^y H_{k,x}[f(x)] dx = y \int_0^\infty u^{-1} \beta(u) F_k(u, y) du, \qquad y > 0,\ k \geqq 3.$$

The function $u^{-1}\beta(u)$ satisfies the hypotheses of Theorem 9.3, and satisfies (9.5) for every positive u. Therefore

$$\lim_{k\to\infty}\int_{0+}^{y} H_{k,x}[f(x)]dx = \beta(y) = \alpha(y) - \alpha(0+), \qquad y > 0.$$

11. **The saltus operator.** We make the following definition:

DEFINITION 11.1. *An operator $h_{k,x}[f(x)]$ is defined by*

$$h_{k,x}[f(x)] = x(8\pi/k)^{1/2}H_{k,x}[f(x)].$$

THEOREM 11.1. *Under the hypotheses of Theorem* 10.2,

$$\lim_{k\to\infty} h_{k,x}[f(x)] = \alpha(x+) - \alpha(x-), \qquad x > 0. \tag{11.1}$$

We consider first the point $x=1$. Introduce the functions

$$\psi(u) = \begin{cases} \alpha(1-), & u < 1, \\ \alpha(1), & u = 1, \\ \alpha(1+), & u > 1, \end{cases}$$

$$\omega(u) = \alpha(u) - \psi(u);$$

$\omega(u)$ is continuous at $u=1$, $\omega(1)=0$. Clearly

$$H_{k,1}[f(x)] = \int_{0+}^{\infty} F_k(u, 1)d\omega(u) + \int_{0}^{\infty} F_k(u, 1)d\psi(u).$$

Now

$$\begin{aligned}\int_0^\infty F_k(u, 1)d\psi(u) &= [\alpha(1+) - \alpha(1-)]F_k(1, 1) \\ &\sim [\alpha(1+) - \alpha(1-)]\left(\frac{k}{8\pi}\right)^{1/2}, \qquad k\to\infty,\end{aligned} \tag{11.2}$$

by Lemma 8.4. For $k \geqq 3$,

$$\int_{0+}^{\infty} F_k(u, 1)d\omega(u) = -\left(\int_0^{1-\eta} + \int_{1-\eta}^{1+\eta} + \int_{1+\eta}^{\infty}\right)\omega(u)dF_k(u, 1) = I_1 + I_2 + I_3,$$

where η, $(0<\eta<1)$, is chosen so that $|\omega(u)| < \epsilon/2$ on $(1-\eta, 1+\eta)$, $\epsilon>0$ being arbitrary. Now $\omega(u)$ satisfies the hypotheses of Lemmas 9.1, 9.2, so that we obtain

$$\lim_{k\to\infty}(I_1 + I_3) = 0.$$

By Lemma 7.4, $F_k(u, 1)$ increases on $(1-\eta, 1)$ and decreases on $(1, 1+\eta)$, so that

$$|I_2| \leqq \int_{1-\eta}^{1} |\omega(u)| \, dF_k(u, 1) - \int_{1}^{1+\eta} |\omega(u)| \, dF_k(u, 1)$$

$$\leqq \frac{\epsilon}{2} [2F_k(1, 1) - F_k(1 - \eta, 1) - F_k(1 + \eta, 1)] \sim \epsilon \left(\frac{k}{8\pi}\right)^{1/2}, \quad k \to \infty,$$

by Lemmas 8.3 and 8.4. Thus

$$\limsup_{k\to\infty} \left(\frac{8\pi}{k}\right)^{1/2} \left| \int_{0+}^{\infty} F_k(u, 1) d\omega(u) \right| \leqq \epsilon,$$

and ϵ was arbitrary. Using also (11.2), we obtain (11.1) for $x=1$.

To establish (11.1) for $x=x_0\neq 1$, we set $x=x_0y$, $t=x_0s$, $u=x_0v$ in (10.3); a simple computation gives

$$g(y) \equiv x_0 f(x_0 y) = \int_{0+}^{\infty} \frac{ds}{y+s} \int_{0+}^{\infty} \frac{d\beta(v)}{s+v}, \qquad \beta(v) = \alpha(x_0 v).$$

By what has already been proved,

$$H_{k,1}[g(y)] \sim \left(\frac{k}{8\pi}\right)^{1/2} [\alpha(x_0 +) - \alpha(x_0 -)], \qquad k \to \infty.$$

But

$$H_{k,1}[g(y)] = \int_{0+}^{\infty} F_k(u, 1) d\alpha(x_0 u) = \int_{0+}^{\infty} F_k(x_0^{-1} u, 1) d\alpha(u)$$

$$= x_0 \int_{0+}^{\infty} F_k(u, x_0) d\alpha(u) = x_0 H_{k, x_0}[f(x)].$$

Hence (11.1) is established in general.

Chapter III. Representation of functions by iterated Stieltjes transforms

12. Theorems on linear differential operators. We consider operators of the form

$$(12.1) \qquad L[f(x)] = \sum_{i=0}^{n} p_{n-i}(x) f^{(i)}(x),$$

where $p_{n-i}(x)=B_{n-i}x^i$; the B_i are constants, and $B_0\neq 0$. We shall call an operator of the form (12.1) an *Euler operator* of order n*. In this section we

* Because $L[f(x)]=g(x)$ is an "Euler differential equation." E. L. Ince, *Ordinary Differential Equations*, 1927, p. 141.

collect the properties of Euler operators which we shall need later.

THEOREM 12.1. *If $L[f(x)]$ is an Euler operator of order n, there exists an operator $\bar{L}[f(x)]$, of order n, called the adjoint of $L[f(x)]$, such that for any functions $f(x)$ and $g(x)$ of class C^n,*

$$g(x)L[f(x)] - f(x)\bar{L}[g(x)] = \frac{d}{dx} P[f(x), g(x)], \tag{12.2}$$

where

$$P[f(x), g(x)] = \sum_{i=1}^{n} \sum_{j=0}^{n-i} (-1)^j f^{(n-i-j)}(x) [p_{i-1}(x)g(x)]^{(j)} \tag{12.3}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_{ij} f^{(i)}(x) g^{(j)}(x) x^{i+j+1}, \tag{12.4}$$

where the A_{ij} are constants. Moreover,

$$\bar{L}[f(x)] = \sum_{i=0}^{n} (-1)^i [p_{n-i}(x)f(x)]^{(i)} \tag{12.5}$$

$$= \sum_{i=0}^{n} A_i f^{(i)}(x) x^i, \qquad A_n \neq 0, \tag{12.6}$$

where the A_i are constants; in particular, $\bar{L}[f(x)]$ is an Euler operator.

The formulas (12.3) and (12.5) are the standard expressions.* To reduce them to (12.4) and (12.6), respectively, one carries out the indicated differentiations and collects terms. The details are left to the reader.

THEOREM 12.2. *If $L[f(x)]$ is an Euler operator of order n, if $f(x)$ and $g(x)$ are of class C^n, and if*

$$f^p(x)g^q(x)x^{p+q+1} \to 0, \qquad 0 \leqq p \leqq n-1; 0 \leqq q \leqq n-1, \tag{12.7}$$

as $x \to 0$ and as $x \to \infty$, then

$$\int_{0+}^{\infty} g(x)L[f(x)]dx = \int_{0+}^{\infty} f(x)\bar{L}[g(x)]dx$$

if either integral converges.

Because of (12.2),

$$\int_{\epsilon}^{R} g(x)L[f(x)]dx - \int_{\epsilon}^{R} f(x)\bar{L}[g(x)]dx = P[f(x), g(x)] \Big|_{\epsilon}^{R}, \quad 0 < \epsilon < R < \infty.$$

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* See, for example, E. L. Ince, op. cit., pp. 123–124.

By (12.4) and (12.7),

$$P[f(x), g(x)] = o(1), \qquad x \to 0,\ x \to \infty.$$

THEOREM 12.3. *If $L[f(x)]$ is an Euler operator of order n, a fundamental set of solutions of the differential equation $L[f(x)]=0$ is*

$$x^{a_i},\ x^{a_i}\log x, \cdots, x^{a_i}(\log x)^{b_i-1}, \qquad i = 1, 2, \cdots, j, \tag{12.8}$$

where the a_i are complex constants, the b_i are positive integers, and $\sum_{i=1}^{j} b_i = n$. Conversely, any set of functions of this form determines (except for a constant multiple) an Euler operator of order n, for which the functions form a fundamental set of solutions.

This is essentially a restatement of known results.* We have

$$L[f(x)] = \sum_{i=0}^{n} B_{n-i} x^i f^{(i)}(x). \tag{12.9}$$

If we set $x = e^z$, it is easily verified that

$$x^i f^{(i)}(x) = \left[\prod_{j=0}^{i-1} (D - j)\right] f(e^z),\dagger$$

where D denotes d/dz, so that

$$L[f(x)] = \sum_{i=0}^{n} B_{n-i}\left[\prod_{j=0}^{i-1} (D - j)\right] f(e^z) = \sum_{i=0}^{n} B'_{n-i} D^i f(e^z) = M[f(e^z)].$$

The linear differential equation with constant coefficients, $M[g(z)]=0$, has, as is well known, a fundamental set of solutions

$$e^{a_i z},\ z e^{a_i z}, \cdots, z^{b_i-1} e^{a_i z}, \qquad i = 1, 2, \cdots, j,$$

with

$$\sum_{i=1}^{j} b_i = n,$$

where the a_i are roots, of respective multiplicities b_i, of the algebraic equation $\sum_{i=0}^{n} B'_{n-i} t^i = 0$. Replacing z by $\log x$, we obtain the functions (12.8).

Conversely, let the functions (12.8) be given. There is a polynomial $P(t) = \sum_{i=0}^{n} B'_{n-i} t^i$ having as roots the a_i with multiplicities b_i. We can write $P(t)$ in the form

$$P(t) = \sum_{i=0}^{n} B_{n-i} \prod_{j=0}^{i-1} (t - j). \tag{12.10}$$

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* E. L. Ince, op. cit., pp. 141–142.

† An empty product denotes unity.

Then the operator $L[f(x)]$, defined by (12.9), with the constants B_i of (12.10), will have the functions (12.8) as a fundamental set of solutions, as the first part of the proof shows. Since $P(t)$ is, except for a constant multiple, uniquely defined, $L[f(x)]$ has the same property.

THEOREM 12.4. *If $L[f(x)]$ is an Euler operator of order n, and $f(x, y)$ is of class C^n and homogeneous of order -1, then $L_x[f(x, y)]=\bar{L}_y[f(x, y)]$.*

From the expressions (12.1) and (12.5) for $L[f(x)]$ and $\bar{L}[f(x)]$, we see that it is sufficient to establish the theorem for the special operator $L[f(x)] =x^kf^{(k)}(x)$, $(k=1, 2, \cdots)$, that is, to prove that

$$(-1)^k \frac{\partial^k}{\partial y^k}(y^k f(x, y)) = x^k \frac{\partial^k}{\partial x^k} f(x, y), \qquad k = 1, 2, \cdots, n. \tag{12.11}$$

For $k=1$, (12.11) is

$$-y\frac{\partial f}{\partial y} - f = x\frac{\partial f}{\partial x},$$

which is Euler's theorem for a homogeneous function of order -1.

We proceed by induction; assuming (12.11) for $k-1$, we establish it for k. The function $x^{-k+1}y^kf$ is homogeneous of order zero; applying Euler's theorem, we have

$$\frac{\partial}{\partial y}\frac{y^kf}{x^{k-1}} = -x\frac{\partial}{\partial x}\frac{y^{k-1}f}{x^{k-1}},$$

$$\begin{aligned}\frac{\partial^k}{\partial y^k}\frac{y^kf}{x^{k-1}} &= -x\frac{\partial}{\partial x}\left[\frac{1}{x^{k-1}}\frac{\partial^{k-1}}{\partial y^{k-1}}(y^{k-1}f)\right]\\ &= -x\frac{\partial}{\partial x}\left[\frac{1}{x^{k-1}}(-1)^{k-1}x^{k-1}\frac{\partial^{k-1}f}{\partial x^{k-1}}\right]\\ &= (-1)^kx\frac{\partial^kf}{\partial x^k};\end{aligned}$$

hence (12.11) is established for k.

THEOREM 12.5. *$H_{k,x}[f(x)]$ has as a set of fundamental solutions*

$$x^n,\ x^n\log x, \qquad n = -k, -k+1, \cdots, -1, 0, 1, \cdots, k-2. \tag{12.12}$$

There are $4k-2$ of these functions; they are clearly linearly independent; it is easily verified that $L_{k,x}[f(x)]$ annuls x^n, and transforms $x^n \log x$ into a constant multiple of x^n, $(n=-k, -k+1, \cdots, k-2)$, so that $H_{k,x}[f(x)]$ annuls all the functions (12.12).

COROLLARY 12.5.1. *$H_{k,x}[f(x)]$ is an Euler operator.*

13. **An auxiliary kernel.** We make the following definition:

DEFINITION 13.1. *An operator $Q[f(x)]$ is defined by*

$$Q[f(x)] = x[x^2f'(x)]'' = x^3f'''(x) + 4x^2f''(x) + 2xf'(x). \tag{13.1}$$

It is evident that $Q[f(x)]$ is an Euler operator.

DEFINITION 13.2. *A function $E(x, t)$ is defined by*

$$E(x, t) = Q_x\left[\frac{\log(x/t)}{x - t}\right].$$

LEMMA 13.1. *If $x>0$, $t>0$, then*

$$E(x, t) = 2x\int_0^\infty \frac{u^2du}{(x + u)^3(t + u)^2}. \tag{13.2}$$

We have

$$g(x) \equiv \frac{\log(x/t)}{x - t} = \int_0^\infty \frac{du}{(x + u)(t + u)},$$

$$g'(x) = -\int_0^\infty \frac{du}{(x + u)^2(t + u)} = -\frac{1}{xt} + \int_0^\infty \frac{du}{(x + u)(t + u)^2},$$

$$[x^2g'(x)]'' = 2\int_0^\infty \frac{u^2du}{(x + u)^3(t + u)^2}.$$

LEMMA 13.2. *If $x>0$ is fixed and $n=0, 1, 2, \cdots$, then*

$$\frac{\partial^n}{\partial t^n} E(x, t) = O(t^{-n-2}\log t), \qquad t \to \infty, \tag{13.3}$$

$$\frac{\partial^n}{\partial t^n} E(x, t) = O(t^{-n}), \qquad t \to 0. \tag{13.4}$$

We have, from (13.2),

$$\frac{\partial^n}{\partial t^n}E(x, t) = 2x(-1)^n(n + 1)!\int_0^\infty \frac{u^2du}{(x + u)^3(t + u)^{n+2}},$$

$$t^n\left|\frac{\partial^n}{\partial t^n} E(x, t)\right| \leqq 2x(n + 1)!\int_0^\infty \frac{u^2du}{(x + u)^3(t + u)^2} \tag{13.5}$$
$$= (n + 1)!E(x, t).$$

From (13.2), we see that (13.4) holds for $n=0$; then (13.4) for $n>0$ follows from (13.5). Also,

$$tE(x, t) \leqq 2x \int_0^\infty \frac{du}{(x+u)(t+u)} = 2x \frac{\log (x/t)}{x - t},$$

and (13.3) for $n=0$ follows; we obtain (13.3) for $n>0$ by use of (13.5).

LEMMA 13.3. *For* $k \geqq 3$,

$$H_{k,x}[f(x)] = M_{k,x}\{Q[f(x)]\}, \tag{13.6}$$

where $M_{k,x}[f(x)]$ *is an Euler operator of order* $4k-5$.

Simple computation shows that, for any real n,

$$Q[x^n] = n^2(n+1)x^n,$$
$$Q[x^n \log x] = n^2(n+1)x^n \log x + (3n^2 + 2n)x^n.$$

Hence $Q[f(x)]$, applied to the functions (12.12), gives either zero or a linear combination of the functions (12.12) other than 1, $\log x$, and $x^{-1} \log x$. By Theorem 12.3, there is an Euler operator $M_{k,x}[f(x)]$, of order $4k-5$, having the functions (12.12), other than 1, $\log x$, and $x^{-1}\log x$, as fundamental solutions. Then $M_{k,x}\{Q[f(x)]\}$ annuls all the functions (12.12) and hence differs from $H_{k,x}[f(x)]$ at most by a constant multiple. If this constant is suitably determined, (13.6) follows.

14. **A general representation theorem.** We prove first the following theorem:

THEOREM 14.1. *If* $f(x)$ *is of class* C^∞, *if*

$$f(x) = o(1), \qquad x \to \infty, \tag{14.1}$$

$$f(x) = o(x^{-1}), \qquad x \to 0, \tag{14.2}$$

and if

$$\int_1^x H_{k,t}[f(x)]dt < O(x), \qquad x \to \infty, \tag{14.3}$$

$$\int_x^1 t^{-4k+2} H_{k,t}[f(x)]dt < O(x^{-4k+2}), \qquad x \to 0, \tag{14.4}$$

each of (14.3) *and* (14.4) *holding for an infinite sequence of positive integers* k, *then*

$$Q[f(x)] = \lim_{k\to\infty} \int_{0+}^\infty H_{k,t}[f(x)]E(x, t)dt, \qquad x > 0, \tag{14.5}$$

and, for $n=0, 1, 2, \cdots$,

$$f^{(n)}(x) = o(x^{-n}), \qquad x \to \infty, \tag{14.6}$$

$$f^{(n)}(x) = o(x^{-n-1}), \qquad x \to 0. \tag{14.7}$$

Let k be an integer greater than unity for which (14.3) holds. Since $H_{k,x}[f(x)]$ is an Euler operator, we may apply a result of R. P. Boas,* in virtue of which (14.1) and (14.3) imply (14.6) for $n=1, 2, \cdots, 4k-4$. Since (14.3) holds for infinitely many k, (14.6) holds for all n. Similarly, (14.4) and (14.2) imply (14.7) for $n=1, 2, \cdots$. Since $f(x)$ satisfies (14.6), (14.7), the function $Q[f(x)]$ also satisfies (14.6), (14.7) (see (13.1)). With Lemma 13.2, these relations imply

$$t^{p+q+1} \frac{d^p}{dt^p} Q[f(t)] \frac{\partial^q}{\partial t^q} E(x, t) = o(1)$$

for $t \to 0$, $t \to \infty$, $p \geqq 0$, $q \geqq 0$. Then by Theorem 12.2,

$$\int_{0+}^{\infty} E(x, t) M_{k,t}\{Q[f(t)]\} dt = \int_{0+}^{\infty} Q[f(t)] \overline{M}_{k,t}[E(x, t)] dt, \tag{14.8}$$

if either integral converges. But $E(x, t)$ is homogeneous of order -1; by Theorem 12.4, Definition 13.2, Lemma 13.3, and Corollary 6.1.1, we have

$$\begin{aligned} \overline{M}_{k,t}[E(x, t)] &= M_{k,x}[E(x, t)] \\ &= M_{k,x}\left\{Q_x\left[\frac{\log (x/t)}{x - t}\right]\right\} \\ &= H_{k,x}\left[\frac{\log (x/t)}{x - t}\right] \\ &= F_k(t, x), \qquad k \geqq 2. \end{aligned}$$

Hence (14.8) becomes

$$\int_{0+}^{\infty} E(x, t) H_{k,t}[f(x)] dt = \int_{0}^{\infty} Q[f(t)] F_k(t, x) dt, \tag{14.9}$$

where the right-hand integral converges for $k \geqq 3$, by Lemma 7.3 and the inequalities satisfied by $Q[f(t)]$. But $Q[f(t)]$ satisfies the conditions of Theorem 9.3 and is continuous for $t > 0$; hence

$$\lim_{k \to \infty} \int_0^{\infty} Q[f(t)] F_k(t, x) dt = Q[f(x)], \qquad x > 0.$$

With (14.9) this yields (14.5).

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* R. P. Boas, *Asymptotic relations for derivatives*, Duke Mathematical Journal, vol. 3 (1937), pp. 637–646, Theorem 2, with $\phi(x) = x\theta(x) = x$, and Theorem 3, with $\phi(x) = x^{-1}\theta(x) = x^{-2}$.

15. **The iterated Stieltjes transform with non-decreasing determining function.** We make the following definition:

DEFINITION 15.1. *A function $f(x)$ will be said to satisfy Conditions* A *if and only if*

(i) *$f(x)$ is of class C^∞ on $(0, \infty)$;*
(ii) *$f(x)=o(1)$, $(x\to\infty)$; $f(x)=o(x^{-1})$, $(x\to 0)$;*
(iii) *for an infinite sequence of positive integers k,*

$$H_{k,x}[f(x)] \geqq 0, \qquad 0 < x < \infty.$$

THEOREM 15.1. *Conditions* A *are necessary and sufficient for $f(x)$ to have the representation*

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u} \tag{15.1}$$

with $\alpha(u)$ normalized and non-decreasing.

That Conditions A, (i) and A, (ii) are satisfied if $f(x)$ has the form (15.1), we know by Theorem 4.1; Condition A, (iii) follows from

$$H_{k,x}[f(x)] = \int_{0+}^{\infty} F_k(u, x)d\alpha(u), \qquad k \geqq 2,$$

since $F_k(u, x) \geqq 0$.

If $f(x)$ satisfies Conditions A, $-f(x)$ satisfies the hypotheses of Theorem 14.1, so that

$$Q[f(x)] = \lim_{k\to\infty} \int_{0+}^{\infty} H_{k,t}[f(x)]E(x, t)dt, \qquad x > 0. \tag{15.2}$$

Formula (13.2) shows that $E(x, t)$ is a positive decreasing function of t, $(0<t<\infty)$; and $E(x, x)=1/(6x)$ (by Lemma 7.1). We then have, since $H_{k,t}[f(x)]\geqq 0$,

$$\int_{0+}^{\infty} H_{k,t}[f(x)]E(x, t)dt \geqq \int_{0+}^{x} H_{k,t}[f(x)]E(x, t)dt \geqq E(x, x)\int_{0+}^{x} H_{k,t}[f(x)]dt,$$

$$\alpha_k(x) \equiv \int_{0+}^{x} H_{k,u}[f(x)]du \leqq 6x\int_{0+}^{\infty} H_{k,t}[f(x)]E(x, t)dt.$$

We define $\alpha_k(0)=0$. If we refer to the proof of Theorem 14.1, we then have, by relation (14.9),

$$\alpha_k(x) \leqq 6x\int_{0}^{\infty} Q[f(t)]F_k(t, x)dt, \qquad x > 0. \tag{15.3}$$

But by (14.6), (14.7), there is a constant A such that

$$Q[f(t)] \leqq A(1 + t^{-1}), \qquad 0 < t < \infty;$$

hence, using (15.3) and Lemma 7.2, we have

$$\begin{aligned} \alpha_k(x) &\leqq 6Ax \int_0^\infty (1 + t^{-1}) F_k(t, x) dt \\ &= 6A \left\{ x + \left(\frac{k-1}{k} \right)^2 \right\}; \end{aligned}$$

and

(15.4) $$\alpha_k(x) < 6A(x + 1), \qquad 0 < x < \infty,$$

where A is independent of k.

The functions $\alpha_k(x)$ (for k is the sequence of Conditions A) are non-decreasing and are bounded, uniformly with respect to k, in each interval $(0, n)$, $(n = 1, 2, \cdots)$. By a theorem of E. Helly,* we can select a subsequence converging in $(0, 1)$, a further subsequence converging in $(0, 2)$, and so on; by use of the diagonal process, we then obtain a subsequence $\{\alpha_{k_i}(x)\}$, converging in $(0, \infty)$ to a non-decreasing function $\alpha(x)$. The relation in (15.2) states that

$$Q[f(x)] = \lim_{k\to\infty} \int_{0+}^\infty E(x, t) d\alpha_k(t) = \lim_{k\to\infty} \int_0^\infty E(x, t) d\alpha_k(t).$$

By use of (13.4), (15.4), and the Helly-Bray theorem,† it follows easily that

$$Q[f(x)] = \int_0^\infty E(x, t) d\alpha(t),$$

or that

(15.5) $$\begin{aligned} [x^2 f'(x)]'' &= 2\int_0^\infty d\alpha(t) \int_0^\infty \frac{u^2 du}{(x+u)^3 (t+u)^2} \\ &= 2\int_0^\infty \frac{t\psi(t) dt}{(x+t)^3}, \\ \psi(t) &= t\int_0^\infty \frac{d\alpha(u)}{(t+u)^2}. \end{aligned}$$

The changes of order of integration, here and for the remainder of the proof,

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* E. Helly, *Über lineare Funktionaloperationen*, Sitzungsberichte der Akademie der Wissenschaften, Vienna, vol. 121 (1921), p. 265.

† See, for example, G. C. Evans, *The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems*, American Mathematical Society Colloquium Publications, vol. 6, New York, 1927, p. 15.

are legitimate because the integrands are positive and $\alpha(t)$ is non-decreasing.*

Since $[x^2f'(x)]'=o(1)$, $(x\to\infty)$, we may integrate (15.5) on (x, ∞), obtaining

$$\begin{aligned}[x^2f'(x)]' &= -2\int_x^\infty dy\int_0^\infty \frac{t\psi(t)dt}{(y+t)^3}\\ &= -\int_0^\infty \frac{t\psi(t)dt}{(x+t)^2}.\end{aligned}$$

Since $x^2f'(x)=o(1)$, $(x\to 0)$, we may now integrate on $(0+, x)$ obtaining

$$\begin{aligned}x^2f'(x) &= -\int_{0+}^x dy\int_0^\infty \frac{t\psi(t)dt}{(y+t)^2}\\ &= -x\int_0^\infty \frac{\psi(t)dt}{x+t}.\end{aligned}$$

The convergence of this integral implies, by use of Lemma 2.3, the convergence of

$$\begin{aligned}\int_t^\infty u^{-1}\psi(u)du &= \int_t^\infty du\int_0^\infty \frac{d\alpha(v)}{(u+v)^2}\\ &= \int_0^\infty \frac{d\alpha(u)}{t+u}.\end{aligned}$$

Then we have

$$\begin{aligned}xf'(x) &= \int_0^\infty \frac{t}{x+t}\,d_t\left(\int_0^\infty \frac{d\alpha(u)}{t+u}\right)\\ &= \frac{-\alpha(0+)}{x} - x\int_0^\infty \frac{dt}{(x+t)^2}\int_0^\infty \frac{d\alpha(u)}{t+u},\end{aligned}$$

since

$$\phi(t) = \int_0^\infty \frac{d\alpha(u)}{t+u}$$

has the properties $\phi(\infty)=0$, and $\phi(t)\sim\alpha(0+)/t$, $(t\to 0)$.† Thus

$$f'(x) = \frac{-\alpha(0+)}{x^2} - \int_0^\infty \frac{dt}{(x+t)^2}\int_0^\infty \frac{d\alpha(u)}{t+u};$$

since $f(\infty)=0$, we may integrate on (x, ∞), obtaining

* The theorem which we use here is the analogue for Stieltjes integrals of the Fubini theorem for Lebesgue integrals; see S. Saks, *Theory of the Integral*, Monografie Matematyczne, vol. 7, Warsaw, 1937, p. 77.

† D. V. Widder, paper cited in §3, p. 10. By the way in which $\alpha(t)$ was defined, we have $\alpha(0)=0$.

$$f(x) = \frac{\alpha(0+)}{x} + \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0}^{\infty} \frac{d\alpha(u)}{t+u};$$

since $f(x) = o(x^{-1})$, $(x \to 0)$, and since the integral has the same property, $\alpha(0+) = \alpha(0) = 0$, and the proof is complete. If $\alpha(t)$ were not normalized, normalization would not affect the representation (15.1); actually, because of Theorem 10.2, we see that our construction yields a normalized function $\alpha(t)$.

THEOREM 15.2. *Conditions* A, *and the additional condition*

$$f(x) = O(x^{-1} \log x), \qquad x \to \infty, \tag{15.6}$$

are necessary and sufficient for $f(x)$ *to have the representation*

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u} \tag{15.7}$$

with $\alpha(t)$ *normalized, non-decreasing, and bounded, on* $(0, \infty)$.

If $f(x)$ has the representation in question, Conditions A are satisfied because of Theorem 15.1. To establish (15.6), we change the order of integration in (15.7)* and write

$$\begin{aligned} f(x) &= \left(\int_{0+}^{1} + \int_{1}^{\infty} \right) \frac{\log (x/t)}{x-t} \, d\alpha(t), \qquad x > 1, \\ &= f_1(x) + f_2(x). \end{aligned}$$

Then

$$\begin{aligned} f_1(x) &\leqq \frac{1}{x-1} \int_{0+}^{1} \log (x/t) d\alpha(t) \\ &= \frac{\log x}{x-1} [\alpha(1) - \alpha(0+)] - \frac{1}{x-1} \int_{0+}^{1} \log t \, d\alpha(t) \\ &= O(x^{-1} \log x), \qquad x \to \infty, \\ f_2(x) &\leqq \frac{\log x}{x-1} \int_{1}^{\infty} d\alpha(t) = O(x^{-1} \log x), \qquad x \to \infty. \end{aligned}$$

Conversely, if $f(x)$ satisfies Conditions A, $f(x)$ has the representation (15.7), and it remains to show that (15.6) implies that $\alpha(t)$ is bounded. Now (15.6) implies that for some constant M,

$$\limsup_{x \to \infty} \frac{x f(x)}{\log x} \leqq M. \tag{15.8}$$

* See the last footnote but one.

We may change the order of integration in (15.7), obtaining

$$f(x) = \int_{0+}^{\infty} \frac{\log (x/t)}{x - t}\, d\alpha(t);$$

since $\alpha(t)$ is non-decreasing and $(\log x - \log t)/(x-t)$ is a positive decreasing function of t, we have, for any $R>0$,

$$\begin{aligned} f(x) &\geqq \int_{0+}^{R} \frac{\log (x/t)}{x - t}\, d\alpha(t) \\ &\geqq \frac{\log (x/R)}{x - R} \int_{0+}^{R} d\alpha(t) = [\alpha(R) - \alpha(0+)] \frac{\log (x/R)}{x - R}; \end{aligned}$$

hence

$$\frac{xf(x)}{\log x} \geqq [\alpha(R) - \alpha(0+)] \frac{\log (x/R)}{\log x} \frac{x}{x - R}. \tag{15.9}$$

If $\alpha(t)$ were unbounded, we could choose R so large that $\alpha(R)-\alpha(0+)>2M$ and then obtain from (15.9)

$$\liminf_{x\to\infty} \frac{xf(x)}{\log x} \geqq 2M,$$

which would contradict (15.8). Hence $\alpha(t)$ is bounded.

16. **A Tauberian theorem.** Representation theorems for the iterated Stieltjes transform with determining function in a class other than that of non-decreasing functions are less easily established than the theorems of §15; there are no available theorems on change of order of integration to carry us, in general, from

$$Q[f(x)] = \int_0^{\infty} E(x, t) d\alpha(t)$$

to

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x + t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t + u}.$$

We shall use, instead, certain consequences of a Tauberian theorem of Hardy and Littlewood,* which we quote as a lemma.

LEMMA 16.1. *If $\phi(t)$ is integrable on every (ϵ, R), $(0<\epsilon<R<\infty)$, if*

$$h(x) = \int_{0+}^{\infty} \frac{\phi(t)dt}{(x + t)^{\rho}} \sim \frac{H}{x^{\sigma}}, \qquad x \to \infty, \tag{16.1}$$

* G. H. Hardy and J. E. Littlewood, *Notes on the theory of series* (XI): *on Tauberian theorems*, Proceedings of the London Mathematical Society, (2), vol. 30 (1930), pp. 23–37; 33.

with $0<\sigma\leqq\rho$, $H\neq 0$, *and if, for almost all* t *not less than some* t_0,

$$\phi(t) \geqq -Kt^{\rho-\sigma-1}, \qquad K > 0; \tag{16.2}$$

then

$$\Phi(t) = \int_{0+}^{t} \phi(u)du \sim \frac{H\Gamma(\rho)}{\Gamma(\sigma)\Gamma(\rho-\sigma+1)} t^{\rho-\sigma}, \qquad t \to \infty. \tag{16.3}$$

The theorem remains true when $H=0$ *if* $\rho>\sigma$, *and* (16.1), (16.3) *are interpreted as* $h(x)=o(x^{-\sigma})$, $\Phi(t)=o(t^{\rho-\sigma})$, *respectively.*

We have modified the original statement of the theorem somewhat, but the modifications are unimportant in their effect.

We shall need also the theorem resulting from one case ($\rho=\sigma=1$) of Lemma 16.1 by the substitutions $t=u^{-1}$, $y=x^{-1}$. For convenience, we state it as the following lemma:

LEMMA 16.2. *If the Stieltjes transform*

$$g(y) = \int_{0+}^{\infty} \frac{\psi(u)du}{y+u}$$

converges, if $\psi(u)>-M$, $(M>0)$, *for almost all* u, $(0<u<\infty)$, *and if*

$$\lim_{y\to 0+} g(y) = G, \qquad G \neq 0,$$

then

$$\int_{0+}^{\infty} u^{-1}\psi(u)du = G.$$

We use Lemmas 16.1, 16.2 to establish the following theorem:

THEOREM 16.3. *Let* $\psi(t)$ *be integrable on every* (ϵ, R), $(0<\epsilon<R<\infty)$, *let*

$$\psi(t) = O(1), \qquad t \to \infty, \tag{16.4}$$

$$\psi(t) = O(t^{-1}), \qquad t \to 0, \tag{16.5}$$

and let

$$\lim_{t\to 0+} t\int_{t}^{1} u^{-1}\psi(u)du \tag{16.6}$$

exist. Assume that $f(x)$, *of class* C^{∞}, *satisfies*

$$f^{(n)}(x) = \begin{cases} o(x^{-n}), & x \to \infty, \\ o(x^{-n-1}), & x \to 0, \end{cases} \tag{16.7}$$

for $n=0, 1, 2, 3$, *and let*

$$[x^2 f'(x)]'' = 2\int_0^\infty \frac{t\psi(t)dt}{(x+t)^3}, \qquad 0 < x < \infty. \tag{16.8}$$

Then

$$\theta(t) = \int_t^\infty u^{-1}\psi(u)du, \qquad t > 0,$$

is defined, and

$$f(x) = \int_{0+}^\infty \frac{\theta(t)}{x+t}\,dt, \qquad 0 < x < \infty.$$

We integrate (16.8) on (x, y), $(0<x<y)$; application of Lemma 2.3 shows that the integral converges uniformly on (x, y); therefore

$$\begin{aligned}[y^2 f'(y)]' - [x^2 f'(x)]' &= \int_0^\infty t\psi(t)\,[(x+t)^{-2} - (y+t)^{-2}]dt \\ &= (y-x)\int_0^\infty \frac{t\psi(t)(x+y+2t)dt}{(x+t)^2(y+t)^2}.\end{aligned} \tag{16.9}$$

As $y\to\infty$, $[y^2f'(y)]'\to 0$, by (16.7). By use of (16.4) it is easily shown that we have

$$x(y-x)\int_0^\infty \frac{t\psi(t)dt}{(x+t)^2(y+t)^2} \to 0, \qquad y\to\infty.$$

By (16.7) and (16.8)

$$\int_0^\infty \frac{t\psi(t)dt}{(x+t)^3} = o(x^{-1}), \qquad x\to\infty;$$

by (16.4), $|t\psi(t)| < Mt$, $(t>1)$, for some constant M. By Lemma 16.1, with $\rho=3$, $\sigma=1$, $H=0$,

$$\gamma(t) \equiv \int_0^t u\psi(u)du = o(t^2), \qquad t\to\infty.$$

Then

$$\begin{aligned}\sigma(t) &\equiv \int_0^t \frac{u^2}{(x+u)^2}\psi(u)du = \int_0^t \frac{u}{(x+u)^2}d\gamma(u) \\ &= \frac{t\gamma(t)}{(x+t)^2} - \int_0^t \gamma(u)\frac{x-u}{(x+u)^3}du \\ &= o(t), \qquad t\to\infty.\end{aligned}$$

Hence

$$y\int_0^\infty \frac{t^2\psi(t)dt}{(x+t)^2(y+t)^2} = y\int_0^\infty \frac{d\sigma(t)}{(y+t)^2} = 2y\int_0^\infty \frac{\sigma(t)dt}{(y+t)^3} = o(1), \quad y\to\infty.*$$

Collecting results, we find from (16.9) that

$$\int_0^\infty \frac{t\psi(t)dt}{(x+t)^2(y+t)^2} \sim -y^{-2}[x^2f'(x)]', \qquad y\to\infty;$$

by (16.4), for fixed $x>0$ and some constant M,

$$\left|\frac{t\psi(t)}{(x+t)^2}\right| < \frac{M}{t}, \qquad t>1.$$

It follows easily from (16.8) that $f(x)$ is analytic. The relation $[x^2f'(x)]'\equiv 0$ is impossible (except in the trivial case $f(x)\equiv 0$, which we exclude from further consideration) because (16.7) excludes all linear combinations (with constant coefficients, not all zero) of the fundamental solutions 1 and x^{-1} of $[x^2f'(x)]' = 0$; hence $[x^2f'(x)]'=0$ at most on a set S of isolated points. For x not in S, we can apply Lemma 16.1, with $\rho=\sigma=2$, obtaining

$$\int_0^\infty \frac{t\psi(t)dt}{(x+t)^2} = -[x^2f'(x)]'.$$

This holds for x in S as well, by continuity.

We integrate this relation on (y, x), $(0<y<x)$, and obtain

$$x^2f'(x) - y^2f'(y) = (y-x)\int_0^\infty \frac{t\psi(t)dt}{(x+t)(y+t)}.$$

As $y\to 0$ we have, by (16.7), $y^2f'(y)=o(1)$ and thus

$$\int_0^\infty \frac{t\psi(t)dt}{(x+t)(y+t)} \to -xf'(x), \qquad y\to 0;$$

also, by (16.4) and (16.5), for fixed $x>0$ and some constant M,

$$\left|\frac{t\psi(t)}{x+t}\right| < M, \qquad t>0.$$

By Lemma 16.2

$$-xf'(x) = \int_{0+}^\infty \frac{\psi(t)dt}{x+t},$$

for all $x>0$ for which $f'(x)\neq 0$; since $f'(x)=0$ at most at a set of isolated points, this relation holds, by continuity, for all $x>0$.

* We leave to the reader the proof of the simple Abelian theorem used here.

A simple application of Lemma 2.3 shows that $\theta(t)$ is defined; we then have

$$- xf'(x) = - \int_{0+}^{\infty} \frac{t}{x+t} d\theta(t)$$
$$= \frac{A}{x} + x \int_{0+}^{\infty} \frac{\theta(t)}{(x+t)^2} dt,$$

where $A = \lim_{t \to 0+} t\theta(t)$ is defined because (16.6) exists. We now have

$$f'(x) = - \frac{A}{x^2} - \int_{0+}^{\infty} \frac{\theta(t)}{(x+t)^2} dt.$$

Integrating on (x, y), $(0<x<y)$, we have

$$f(x) - f(y) = A\left(\frac{1}{x} - \frac{1}{y}\right) + (y - x) \int_{0+}^{\infty} \frac{\theta(t)dt}{(x+t)(y+t)},$$

$$\int_{0+}^{\infty} \frac{\theta(t)dt}{(x+t)(y+t)} \sim \frac{1}{y}\left[f(x) - \frac{A}{x}\right], \qquad y \to \infty,$$

and

$$\left| \frac{\theta(t)}{x+t} \right| \leqq \frac{M}{t}, \qquad 0 < t < \infty,$$

for some constant M. By Lemma 16.1 (with $\rho = \sigma = 1$),

$$f(x) - \frac{A}{x} = \int_{0+}^{\infty} \frac{\theta(t)}{x+t} dt$$

for all $x>0$ for which $f(x) - Ax^{-1} \neq 0$, and hence (by continuity) for all $x>0$, since (16.7) excludes $f(x) = Ax^{-1}$. But $f(x) = o(x^{-1})$, $(x \to 0)$; and by (4.5) we obtain

$$\int_{0+}^{\infty} \frac{\theta(t)}{x+t} dt = o(x^{-1}), \qquad x \to 0.$$

Hence $A = 0$, and the proof is complete.

17. **Determining function of bounded variation on** $(0, \infty)$. We introduce the following definition:

DEFINITION 17.1. *A function* $f(x)$ *is said to satisfy Conditions* B *if and only if*

(i) $f(x)$ *is of class* C^∞ *on* $(0, \infty)$;
(ii) $f(x) = o(x^{-1})$, $(x \to 0)$; $f(x) = o(1)$, $(x \to \infty)$;
(iii) *for an infinite sequence of positive integers* k,

$$\int_0^\infty |H_{k,t}[f(x)]|\,dt < M,$$

where M is independent of k.

THEOREM 17.1. *Conditions* B *are necessary and sufficient for $f(x)$ to have the representation*

$$f(x) = \int_{0+}^\infty \frac{dt}{x+t}\int_{0+}^\infty \frac{d\alpha(u)}{t+u}, \tag{17.1}$$

where $\alpha(u)$ is a normalized function of bounded variation on $(0, \infty)$.

If $f(x)$ has the form (17.1), Conditions B, (i) and B, (ii) are satisfied, by Theorem 4.1. As for Condition B, (iii), by Theorem 6.1

$$\begin{aligned} |H_{k,t}[f(x)]| &= \left|\int_0^\infty F_k(u, t)d\alpha(u)\right| \\ &\leqq \int_0^\infty F_k(u, t)\,|d\alpha(u)|, \qquad k \geqq 2; \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty |H_{k,t}[f(x)]|\,dt &\leqq \int_0^\infty dt \int_0^\infty F_k(u, t)\,|d\alpha(u)| \\ &= \int_0^\infty |d\alpha(u)| \int_0^\infty F_k(u, t)dt^* \\ &= \int_0^\infty |d\alpha(u)| = M. \end{aligned}$$

To establish the converse we apply Theorem 14.1. Conditions (14.1), (14.2), and (14.3) are evidently satisfied. To establish (14.4) we write

$$\alpha_k(t) = \int_v^t H_{k,u}[f(x)]du, \qquad t \geqq 0; \tag{17.2}$$

then

$$\begin{aligned} \int_x^1 t^{-4k+2}H_{k,t}[f(x)]dt &= \int_x^1 t^{-4k+2}d\alpha_k(t) \\ &= \alpha_k(1) - \alpha_k(x)x^{-4k+2} + (4k-2)\int_x^1 \alpha_k(t)t^{-4k+1}dt \\ &= o(x^{-4k+2}), \qquad x \to 0, \end{aligned}$$

* We have again used the Stieltjes analogue of the Fubini theorem. See the third footnote in §15.

since $\alpha_k(t)=o(1)$, $(t\to 0)$; this holds for the sequence of integers k of Condition B, (iii). Theorem 14.1 now gives

$$Q[f(x)] = \lim_{k\to\infty} \int_0^\infty E(x, t)d\alpha_k(t), \tag{17.3}$$

where $\alpha_k(t)$ is defined by (17.2), and we think of k as restricted to the sequence of Condition B, (iii); Theorem 14.1 also shows that relations (14.6), (14.7) are satisfied.

Condition B, (iii) states that the functions $\alpha_k(t)$ have uniformly bounded variation on $(0, \infty)$. By Helly's theorem* we can pick a subsequence $\{\alpha_{k_i}(t)\}$ converging to a function $\alpha(t)$ of bounded variation on $(0, \infty)$. The function $E(x, t)$ is continuous on $(0, \infty)$ and approaches zero as $t\to\infty$; it follows easily from the Helly-Bray theorem† that we may take the limit under the integral sign, over the sequence $\{k_i\}$, in (17.3). That is,

$$\begin{aligned} Q[f(x)] &= \int_0^\infty E(x, t)d\alpha(t) \\ &= \int_{0+}^\infty E(x, t)d\alpha(t), \end{aligned}$$

and

$$\int_0^\infty |d\alpha(t)| \leqq M.$$

Using the expressions (13.1), (13.2) for $Q[f(x)]$ and $E(x, t)$, we obtain

$$\begin{aligned} [x^2f'(x)]'' &= 2\int_{0+}^\infty d\alpha(t) \int_0^\infty \frac{u^2du}{(x+u)^3(t+u)^2} \\ &= 2\int_0^\infty \frac{t\psi(t)dt}{(x+t)^3}, \end{aligned} \tag{17.4}$$

$$\psi(t) = t\int_{0+}^\infty \frac{d\alpha(u)}{(t+u)^2}.\ddagger \tag{17.5}$$

We now apply Theorem 16.3. Clearly (16.4) and (16.5) are satisfied; also

$$g(t) = \int_{0+}^\infty \frac{d\alpha(u)}{t+u}$$

exists because $\alpha(t)$ has bounded variation on $(0, \infty)$, $g(t)=o(t^{-1})$, $(t\to 0)$, (by Theorem 4.1), and

* E. Helly, loc. cit.

† G. C. Evans, loc. cit.

‡ For the change of order of integration, see the third footnote in §15.

$$\psi(t) = -tg'(t),$$

so that the limit (16.6) exists. Conditions (16.7) are contained in (14.6), (14.7), which we saw above to be consequences of Conditions B, (ii) and B, (iii), through Theorem 14.1. Then by Theorem 16.3,

$$\begin{aligned} f(x) &= \int_{0+}^{\infty} \frac{dt}{x+t} \int_t^{\infty} \frac{\psi(u)}{u} du \\ &= -\int_{0+}^{\infty} \frac{dt}{x+t} \int_t^{\infty} g'(u) du \\ &= \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u} \end{aligned}$$

(where we have used the fact that $g(\infty)=0$). By the definition of $\alpha(t)$, we obtain $\alpha(0)=0$; that $\alpha(t)$ is normalized follows from Theorem 10.2 and (17.2), because of the way in which $\alpha(t)$ was defined.

18. **Determining function the integral of a function of L^p, $(p>1)$.** We make the following definition:

DEFINITION 18.1. *A function $f(x)$ satisfies Conditions* C *if and only if*

(i) *$f(x)$ is of class C^∞ on* $(0, \infty)$;

(ii) $f(x)=o(1)$, $(x\to\infty)$; $f(x)=o(x^{-1})$, $(x\to 0)$;

(iii) *for an infinite sequence of positive integers k,*

$$\int_0^{\infty} |H_{k,t}[f(x)]|^p dt < M, \qquad p > 1,$$

where M is independent of k.

THEOREM 18.1. *Conditions* C *are necessary and sufficient for $f(x)$ to have the representation*

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{\phi(u)du}{t+u}, \tag{18.1}$$

with $\phi(u)$ of class L^p on $(0, \infty)$, $(p>1)$.

If $f(x)$ has the form (18.1), Conditions C, (i) and C, (ii) are satisfied, by Theorem 4.1. As for Condition C, (iii),

$$H_{k,x}[f(x)] = \int_0^{\infty} F_k(u, x)\phi(u)du;$$

then, by use of Hölder's inequality and the Fubini theorem,

$$\begin{aligned}
\left(\frac{k-1}{k}\right)^2 \int_0^\infty |\phi(u)|^p du &= \int_0^\infty |\phi(u)|^p du \int_0^\infty F_k(u, x)dx \\
&= \int_0^\infty dx \int_0^\infty F_k(u, x) |\phi(u)|^p du \\
&= \int_0^\infty \left\{\left(\int_0^\infty F_k(u, x)du\right)^{p/q}\right. \\
&\qquad \cdot \left.\int_0^\infty F_k(u, x) |\phi(u)|^p du\right\} dx, \qquad 1/p + 1/q = 1, \\
&\geqq \int_0^\infty \left(\int_0^\infty F_k(u, x)^{1/q} F_k(u, x)^{1/p} |\phi(u)| \, du\right)^p dx \\
&= \int_0^\infty \left(\int_0^\infty F_k(u, x) |\phi(u)| \, du\right)^p dx \\
&\geqq \int_0^\infty \left|\int_0^\infty F_k(u, x)\phi(u)du\right|^p dx \\
&= \int_0^\infty |H_{k,x}[f(x)]|^p dx, \qquad k = 2, 3, \cdots .
\end{aligned}$$

We may take $M = \int_0^\infty |\phi(u)|^p du$.

To establish the converse, we apply Theorem 14.1. We need only verify (14.3) and (14.4). To do this, we have, for the sequence of integers of Condition C, (iii) by Hölder's inequality,

$$\left|\int_1^x H_{k,t}[f(x)]dt\right| \leqq x^{1/q}\left(\int_0^x |H_{k,t}[f(x)]|^p dt\right)^{1/p} = o(x), \qquad x \to \infty,$$

$$\begin{aligned}
\left|\int_x^1 t^{-4k+2} H_{k,t}[f(x)]dt\right| &\leqq x^{-4k+2+1/q}\left(\int_x^\infty |H_{k,t}[f(x)]|^p dt\right)^{1/p} \\
&= o(x^{-4k+2}), \qquad x \to 0.
\end{aligned}$$

Then by Theorem 14.1,

$$Q[f(x)] = \lim_{k\to\infty} \int_0^\infty H_{k,t}[f(x)]E(x, t)dt,$$

with k in the sequence in question; and (14.6), (14.7) are satisfied. By the weak compactness of the space L^p,* there is a function $\phi(t)$ of L^p, such that for every function $\omega(t)$ of L^q, $(1/p+1/q=1)$,

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* S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 130. Banach gives the theorem in question only for a finite interval, but it is equally valid for the infinite interval.

$$\lim_{k\to\infty}\int_0^\infty H_{k,t}[f(x)]\omega(t)dt = \int_0^\infty \phi(t)\omega(t)dt.$$

It follows from Lemma 13.2 that $E(x, t)$ belongs to every L^q, $(q>1)$; hence

$$Q[f(x)] = \int_0^\infty \phi(t)E(x, t)dt,$$

with $\phi(t)$ belonging to $L^p(0, \infty)$. That is,

$$\begin{aligned}[x^2f'(x)]'' &= 2\int_0^\infty \phi(t)dt\int_0^\infty \frac{u^2du}{(x+u)^3(t+u)^2}\\ &= 2\int_0^\infty \frac{t\psi(t)dt}{(x+t)^3},\\ \psi(t) &= t\int_0^\infty \frac{\phi(u)du}{(t+u)^2};\end{aligned}$$

the change of the order of integration is justified by Fubini's theorem.

We now apply Theorem 16.3. By Hölder's inequality

$$\begin{aligned}|\psi(t)| &\leqq t\left(\int_0^\infty |\phi(u)|^p du\right)^{1/p} t^{-2+1/q}, &\qquad 1/p + 1/q = 1,\\ &= O(t^{-1+1/q}), &\qquad t\to 0,\ t\to\infty;\end{aligned}$$

thus (16.4) and (16.5) are satisfied, and the limit (16.6) exists. Conditions (16.7) are included in (14.6) and (14.7), which we have already established. Moreover,

$$g(t) = \int_{0+}^\infty \frac{\phi(u)du}{t+u}, \qquad t>0,$$

is seen to exist, by another application of Hölder's inequality; $\psi(t) = -tg'(t)$; and $g(\infty)=0$. By Theorem 16.3

$$\begin{aligned}f(x) &= \int_{0+}^\infty \frac{dt}{x+t}\int_t^\infty \frac{\psi(u)}{u}du\\ &= \int_{0+}^\infty \frac{dt}{x+t}\int_{0+}^\infty \frac{\phi(u)du}{t+u},\end{aligned}$$

and the proof is complete.

19. **Determining function the integral of a function of class** L. Theorem 18.1 fails when $p=1$, since for $p=1$, Conditions C reduce to Conditions B. To treat the case $p=1$, we introduce the following definition:

DEFINITION 19.1. *A function $f(x)$ satisfies Conditions* D *if and only if*

(i) *$f(x)$ is of class C^∞ on* $(0, \infty)$;

(ii) $f(x)=o(1)$, $(x\to\infty)$; $f(x)=o(x^{-1})$, $(x\to 0)$;

(iii) *for some infinite sequence of positive integers k, $H_{k,t}[f(x)]$ belongs to $L(0, \infty)$, and for m and n in the sequence,*

$$\lim_{m,n\to\infty}\int_0^\infty | H_{m,t}[f(x)] - H_{n,t}[f(x)] |\, dt = 0.$$

THEOREM 19.1. *Conditions* D *are necessary and sufficient for $f(x)$ to have the representation*

$$f(x) = \int_{0+}^\infty \frac{dt}{x+t}\int_{0+}^\infty \frac{\phi(u)du}{t+u}, \tag{19.1}$$

with $\phi(u)$ integrable on $(0, \infty)$.

If $f(x)$ has the form (19.1), Conditions D, (i) and D, (ii) are certainly satisfied. To verify Condition D, (iii), we note that for $k\geqq 2$

$$\begin{aligned}\left(\frac{k-1}{k}\right)^2\int_0^\infty |\phi(u)|\,du &= \int_0^\infty |\phi(u)|\,du\int_0^\infty F_k(u, t)dt\\ &= \int_0^\infty dt\int_0^\infty |\phi(u)| F_k(u, t)du\\ &\geqq \int_0^\infty | H_{k,t}[f(x)] |\, dt,\end{aligned}$$

so that $H_{k,t}[f(x)]$ belongs to $L(0, \infty)$, $(k\geqq 2)$. In addition

$$\begin{aligned}| H_{k,t}[f(x)] - \phi(t) | &\leqq \int_0^\infty F_k(u, t) | \phi(u) - \phi(t) |\, du\\ &= \int_0^\infty F_k(u, 1) | \phi(ut) - \phi(t) |\, du;\end{aligned}$$

$$\int_0^\infty | H_{k,t}[f(x)] - \phi(t) |\, dt \leqq \int_0^\infty dt\int_0^\infty F_k(u, 1) | \phi(ut) - \phi(t) |\, du,$$

if the iterated integral converges. It will converge if

$$\int_0^\infty F_k(u, 1)g(u)du \tag{19.2}$$

converges, where

$$g(u) = \int_0^\infty | \phi(ut) - \phi(t) |\, dt.$$

But for some constant A

$$g(u) < A(1 + u^{-1});$$

hence (19.2) converges ($k \geqq 3$). Furthermore, $g(u)$ is continuous at $u=1$, and $g(1)=0$.* Corresponding to an arbitrary $\epsilon>0$, we determine δ, $(0<\delta<1)$, so that

$$g(u) < \epsilon, \qquad |u-1| < \delta.$$

Then

$$\int_0^\infty | H_{k,t}[f(x)] - \phi(t) | dt \leqq \left(\int_0^{1-\delta} + \int_{1-\delta}^{1+\delta} + \int_{1+\delta}^\infty \right) F_k(u, 1) g(u) du$$
$$= I_1 + I_2 + I_3;$$

$$I_2 \leqq \epsilon \int_0^\infty F_k(u, 1) du = \epsilon,$$

$$I_1 \leqq A \int_0^{1-\delta} (1 + u^{-1}) F_k(u, 1) du \leqq 2A \int_0^{1-\delta} u^{-1} F_k(u, 1) du$$
$$= 2A \int_0^{1-\delta} F_k(1, u) du$$
$$= 2A H_k(1-\delta) = o(1),\dagger \qquad k \to \infty,$$

$$I_3 \leqq A \int_{1+\delta}^\infty (1 + u^{-1}) F_k(u, 1) du \leqq 2A \int_{1+\delta}^\infty F_k(u, 1) du$$
$$= 2A d_k^2 \int_{1+\delta}^\infty u^k du \int_0^\infty \frac{t^{2k-1} dt}{(u+t)^{2k}(1+t)^{2k}}$$
$$\leqq 2A d_k^2 \int_{1+\delta}^\infty u^{k-2} du \int_0^\infty \frac{t^{2k-3} dt}{(u+t)^{2k-2}(1+t)^{2k-2}}$$
$$= \frac{2A d_k^2}{d_{k-1}^2} \left\{ \left(\frac{k-2}{k-1} \right)^2 - H_{k-1}(1+\delta) \right\}$$
$$= o(1), \qquad k \to \infty.$$

It follows that

$$\lim_{k \to \infty} \int_0^\infty | H_{k,t}[f(x)] - \phi(t) | dt = 0,$$

which implies Condition D, (iii).

* D. V. Widder, *A classification of generating functions*, these Transactions, vol. 39 (1936), p. 267.

† $H_k(1-\delta)$ is the function of Lemma 8.2.

We now establish the sufficiency of our conditions. Condition D, (iii) implies* the existence of a function $\phi(t)$, integrable on $(0, \infty)$, such that

$$\lim_{k\to\infty} \int_0^\infty | H_{k,t}[f(x)] | dt = \int_0^\infty | \phi(t) | dt,$$

$$\lim_{k\to\infty} \int_0^t H_{k,u}[f(x)]du = \int_0^t \phi(u)du, \qquad t > 0,$$

where k runs through the sequence of Condition D, (iii). Consequently,

$$\int_0^\infty | H_{k,t}[f(x)] | dt \leqq \int_0^\infty | \phi(t) | dt + 1$$

for k greater than some k_0, and k in the sequence. Thus $f(x)$ satisfies Conditions B, and by Theorem 17.1

$$f(x) = \int_{0+}^\infty \frac{dt}{x+t} \int_{0+}^\infty \frac{d\alpha(u)}{t+u},$$

with $\alpha(t)$ a normalized function, of bounded variation on $(0, \infty)$. By Theorem 10.2,

$$\alpha(t) - \alpha(0+) = \lim_{k\to\infty} \int_0^t H_{k,u}[f(x)]du, \qquad t > 0.$$

But

$$\lim_{k_i\to\infty} \int_0^t H_{k_i,u}[f(x)]du = \int_0^t \phi(u)du,$$

where $\{k_i\}$ is a certain subsequence of the integers, and consequently

$$\alpha(t) - \alpha(0+) = \int_0^t \phi(u)du.$$

Hence $f(x)$ has the form (19.1).

20. **Determining function the integral of a bounded function.** We introduce the following definition:

Definition 20.1. *A function $f(x)$ satisfies Conditions* E *if and only if*

(i) *$f(x)$ is of class C^∞ on $(0, \infty)$;*

(ii) $f(x) = o(1)$, $(x\to\infty)$; $f(x) = o(x^{-1})$, $(x\to 0)$;

(iii) *for an infinite sequence of positive integers k,*

$$| H_{k,x}[f(x)] | \leqq M, \qquad 0 < x < \infty,$$

where M is independent of k.

* See, for example, E. C. Titchmarsh, *The Theory of Functions*, 1932, pp. 387 ff.

THEOREM 20.1. *Conditions* E *are necessary and sufficient for* $f(x)$ *to have the representation*

$$f(x)=\int_{0+}^{\infty}\frac{dt}{x+t}\int_{0+}^{\infty}\frac{\phi(u)du}{t+u}, \tag{20.1}$$

with $\phi(u)$ *bounded almost everywhere.*

If $f(x)$ has the form (20.1), we have only to verify Condition E, (iii). For $k\geqq 2$ we have

$$\left| H_{k,x}[f(x)]\right| = \left|\int_0^\infty F_k(u,x)\phi(u)du\right| \leqq M\int_0^\infty F_k(u,x)du = M,$$

where $|\phi(u)|\leqq M$ almost everywhere.

To show that the conditions are sufficient, we use Theorem 14.1, whose hypotheses are evidently fulfilled. It follows that

$$Q[f(x)] = \lim_{k\to\infty}\int_0^\infty H_{k,t}[f(x)]E(x,t)dt.$$

By the weak compactness of the space of functions bounded almost everywhere,* there exists a function $\phi(t)$, bounded almost everywhere, such that for every function $\omega(t)$ of $L(0,\infty)$

$$\lim_{k\to\infty}\int_0^\infty H_{k,t}[f(x)]\omega(t)dt = \int_0^\infty \phi(t)\omega(t)dt.$$

Since $E(x,t)$ belongs to $L(0,\infty)$, we obtain

$$[x^2f'(x)]'' = 2\int_0^\infty \phi(t)dt\int_0^\infty \frac{u^2du}{(x+u)^3(t+u)^2} = 2\int_0^\infty \frac{t\psi(t)dt}{(x+t)^3},$$

$$\psi(t) = t\int_0^\infty \frac{\phi(u)du}{(t+u)^2}.$$

We now apply Theorem 16.3. Since

$$|\psi(t)|\leqq M,$$

the conditions of that theorem are satisfied, and

$$f(x) = \int_{0+}^{\infty}\frac{\theta(t)}{x+t}dt, \tag{20.2}$$

$$\theta(x) = \int_x^\infty dt\int_0^\infty \frac{\phi(u)du}{(t+u)^2}.$$

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* S. Banach, loc. cit. Banach gives the theorem only for a finite interval.

For $y>x$ we have

$$\theta(y) - \theta(x) = (x - y)\int_0^\infty \frac{\phi(u)du}{(x+u)(y+u)},$$

since the integral defining $\psi(t)$ is evidently uniformly convergent $(x \leqq t \leqq y)$. Hence

$$\int_0^\infty \frac{\phi(u)du}{(x+u)(y+u)} \sim \frac{\theta(x)}{y} \qquad y \to \infty;$$

and

$$\left| \frac{\phi(u)}{x+u} \right| \leqq \frac{M}{u}$$

for almost all u, and for fixed x. By Lemma 16.1 (with $\rho=\sigma=1$),

$$\theta(t) = \int_0^\infty \frac{\phi(u)du}{t+u}, \tag{20.3}$$

for all $t>0$ for which $\theta(t) \neq 0$. Since $\theta(t)=0$ at most on a set of isolated points (except in the trivial case $f(x) \equiv 0$), (20.3) holds by continuity for all $t>0$. Substitution of this formula for $\theta(t)$ into (20.2) completes the proof.

Chapter IV. The representation of functions by S_2 transforms

21. Determining function non-decreasing; determining function the integral of a function of class L^p, $(p>1)$. In these two cases the S_2 transform and the iterated Stieltjes transform are equivalent. The S_2 transform is obtained from the iterated Stieltjes transform by a formal change of the order of integration. If the determining function is non-decreasing this formal process is legitimate; it is also legitimate if the determining function is the integral of a function of class L^p, $(p>1)$. In fact, if $\phi(t)$ belongs to L^p, $(p>1)$, then

$$\left| \int_0^\infty \frac{\phi(u)du}{t+u} \right| \leqq \left(\int_0^\infty |\phi(u)|^p du \right)^{1/p} \left(\int_0^\infty \frac{du}{(t+u)^q} \right)^{1/q}, \qquad 1/p + 1/q = 1,$$

$$= (q-1)^{-1/q} t^{-1/p} \left(\int_0^\infty |\phi(u)|^p du \right)^{1/p},$$

and since $t^{-1/p}/(x+t)$ is integrable on $(0, \infty)$, $(x>0)$,

$$\int_0^\infty \frac{dt}{x+t} \int_0^\infty \frac{|\phi(u)|du}{t+u}$$

exists and dominates

$$\int_0^\infty \frac{dt}{x+t}\int_0^\infty \frac{\phi(u)du}{t+u}.$$

We may therefore state the following theorems:

THEOREM 21.1. *A necessary and sufficient condition that*

$$f(x) = \int_{0+}^\infty \frac{\log (x/t)}{x-t}\, d\alpha(t),$$

with $\alpha(t)$ normalized and non-decreasing, is that $f(x)$ satisfy Conditions A.

THEOREM 21.2. *A necessary and sufficient condition that*

$$f(x) = \int_{0+}^\infty \frac{\log (x/t)}{x-t}\, \phi(t)dt,$$

with $\phi(t)$ belonging to $L^p(0, \infty)$, $(p>1)$, is that $f(x)$ satisfy Conditions C.

Corresponding to the representation theorems for the iterated Stieltjes transform in the other cases, there are representation theorems for the S_2 transform; in each case an auxiliary condition is imposed to make application of Theorem 5.3 possible.

22. **A lemma.** We can make the following statement:

LEMMA 22.1. *Let*

$$H_k(y) = \int_0^y F_k(1, x)dx.*$$

Then there is a constant A such that

$$(22.1) \qquad y^{-1/2}H_k(y) \leqq A, \qquad 0 < y \leqq 1/2,$$

uniformly with respect to k, $(k \geqq 2)$.

We refer to the proof of Lemma 8.2 (page 23), where we find the relation

$$H_k(y) \leqq 2d_k \int_0^{y^{1/2}} \frac{t^{k-1}dt}{(t+1)^{2k}}.$$

Since $t(t+1)^{-2}$ increases on $(0, y^{1/2})$, we have

$$H_k(y) \leqq 2d_k\left(\frac{y^{1/2}}{(y^{1/2}+1)^2}\right)^{k-1}\int_0^{y^{1/2}} \frac{dt}{(1+t)^2} = 2d_k\left(\frac{y^{1/2}}{(y^{1/2}+1)^2}\right)^{k-1}\frac{y^{1/2}}{1+y^{1/2}}.$$

But

$$d_k = \frac{(2k-1)!}{k!(k-2)!}, \qquad k \geqq 2.$$

* $H_k(y)$ is the function of Lemma 8.2.

By use of Stirling's formula, we see that there is a positive constant B such that

$$d_k \leqq Bk^{1/2}2^{2k}, \qquad k \geqq 2.$$

Since $0<y\leqq 1/2$, there is a constant $\lambda<1$ such that

$$4y^{1/2}(y^{1/2}+1)^{-2} \leqq \lambda.$$

Then

$$H_k(y) \leqq 8Bk^{1/2}\lambda^{k-1}y^{1/2}(1+y^{1/2})^{-1} \leqq Ay^{1/2}, \qquad k \geqq 2,$$

for a suitably chosen constant A.

23. **Determining function of bounded variation on** $(0, \infty)$. We prove the following theorem:

THEOREM 23.1. *A necessary and sufficient condition that $f(x)$ have the representation*

$$f(x) = \int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, d\alpha(t), \tag{23.1}$$

with $\alpha(t)$ a normalized function, of bounded variation on $(0, \infty)$, is that $f(x)$ should satisfy Conditions B, *and that for an infinite sequence of positive integers k,*

$$\log \frac{1}{t}\left| \int_{0+}^{t} H_{k,u}[f(x)]du \right| < \epsilon(t), \qquad 0 < t \leqq 1/4, \tag{23.2}$$

where $\lim_{t\to 0} \epsilon(t)=0$, *and $\epsilon(t)$ is independent of k.*

We show first that if $f(x)$ has the form (23.1), then (23.2) is satisfied. By Theorem 5.2, $f(x)$ is an iterated Stieltjes transform; hence

$$\begin{aligned} H_{k,t}[f(x)] &= \int_0^{\infty} F_k(u, t)d\alpha(u), \qquad k \geqq 2, \\ \int_{0+}^{x} H_{k,t}[f(x)]dt &= \int_{0+}^{x} dt \int_0^{\infty} F_k(u, t)d\alpha(u) = \int_0^{\infty} d\alpha(u) \int_0^{x} F_k(u, t)dt; \end{aligned} \tag{23.3}$$

the change of the order of integration is legitimate because under our hypotheses, the last integral is absolutely convergent.*

The function

$$\int_0^{x} F_k(u, t)dt$$

* See the third footnote in §15.

is a decreasing function of u. For, $tF_k(u, t)$ is homogeneous of order zero,

$$\frac{\partial}{\partial u} F_k(u, t) = -\frac{1}{u}\frac{\partial}{\partial t}(tF_k(u, t)),$$

and

$$\frac{\partial}{\partial u}\int_0^x F_k(u, t)dt = \int_0^x \frac{\partial}{\partial u} F_k(u, t)dt$$
$$= -xu^{-1}F_k(u, x) < 0, \qquad u > 0, x > 0.$$

We now have, from (23.3),

$$\int_{0+}^x H_{k,t}[f(x)]dt = \left(\int_{0+}^{x^{1/2}} + \int_{x^{1/2}}^{\infty}\right) d\alpha(u) \int_0^x F_k(u, t)dt = I_1 + I_2.$$

Using Lemma 2.2, which applies because $\int_0^x F_k(u, t)dt$ is a positive decreasing function of u, we obtain

$$|I_1| \leqq \left(\lim_{u\to 0+}\int_{0+}^x F_k(u, t)dt\right) \underset{0\leqq y\leqq x^{1/2}}{\text{u.b.}} |\alpha(y) - \alpha(0+)|.$$

But

$$\int_0^x F_k(u, t)dt \leqq \int_0^\infty F_k(u, t)dt = \left(\frac{k-1}{k}\right)^2 < 1,$$

and

$$|I_1| \leqq \underset{0\leqq y\leqq x^{1/2}}{\text{u.b.}} |\alpha(y) - \alpha(0+)|. \tag{23.4}$$

Let

$$\int_0^\infty |d\alpha(u)| = M.$$

Since $\int_0^x F_k(u, t)dt$ is a decreasing function of u,

$$|I_2| \leqq M\int_0^x F_k(x^{1/2}, t)dt = Mx^{1/2}\int_0^{x^{1/2}} F_k(x^{1/2}, x^{1/2}u)du, \qquad t = ux^{1/2},$$
$$= M\int_0^{x^{1/2}} F_k(1, u)du = MH_k(x^{1/2}).$$

According to Lemma 22.1, then, there is a constant A such that

$$|I_2| \leqq AMx^{1/4}, \qquad k \geqq 2, 0 < x \leqq 1/4.$$

Combining this with (23.4), we have

$$|I_1 + I_2| \leqq \epsilon(x), \qquad 0 < x \leqq 1/4,$$

where

$$\epsilon(x) = \underset{0 \leqq y \leqq x^{1/2}}{\text{u.b.}} |\alpha(y) - \alpha(0+)| + AMx^{1/4}.$$

The function $\epsilon(x)$ is independent of k, $(k \geqq 2)$; and $\epsilon(x) = o(-1/\log x)$, $(x \to 0)$, since by Theorem 3.1,

$$\underset{0 \leqq y \leqq x^{1/2}}{\text{u.b.}} |\alpha(y) - \alpha(0+)| = o(-1/\log x^{1/2}) = o(-1)/\log x).$$

Conversely let us suppose that $f(x)$ satisfies the conditions of the theorem. Conditions B imply that

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u}, \tag{23.5}$$

with $\alpha(u)$ a normalized function of bounded variation on $(0, \infty)$. By Theorem 10.2,

$$\alpha(u) - \alpha(0+) = \lim_{k\to\infty} \int_{0+}^{u} H_{k,t}[f(x)]dt, \qquad u > 0.$$

Therefore

$$|\alpha(u) - \alpha(0+)| \log(1/u) = \lim_{k\to\infty} \log(1/u) \left| \int_{0+}^{u} H_{k,t}[f(x)]dt \right|.$$

But by (23.2),

$$\log(1/u) \left| \int_{0}^{u} H_{k,t}[f(x)]dt \right| \leqq \epsilon(u), \qquad 0 < u \leqq 1/4,$$

for some sequence of integers k. Hence

$$|\alpha(u) - \alpha(0+)| \log(1/u) \leqq \epsilon(u), \qquad 0 < u \leqq 1/4. \tag{23.6}$$

Furthermore,

$$\int_{0}^{\infty} |d\alpha(u)| = M,$$

and for $t > 0$,

$$\left| \int_{t}^{\infty} \frac{d\alpha(u)}{u} \right| \leqq \frac{M}{t} = o(1/\log t), \qquad t \to \infty. \tag{23.7}$$

Conditions (23.6) and (23.7) are the conditions of Theorem 5.2; since they are satisfied, we may change the order of integration in (23.5) to obtain the representation (23.1) for $f(x)$.

The condition (23.2) appears highly artificial; one might hope to replace it by a weaker condition which, together with Conditions A would still be sufficient for $f(x)$ to have the representation (23.1). This, however, does not appear to be possible. Inequality (23.2) states that

$$\int_{0+}^{x} H_{k,t}[f(x)]dt = o(-1/\log x), \qquad x \to 0,$$

where the function $o(-1/\log x)$ is independent of k; if the uniform $o(-1/\log x)$ is replaced by a uniform $O(-1/\log x)$ and a non-uniform $o(-1/\log x)$, then Theorem 23.1 ceases to be true. This is verified by the following theorem:

THEOREM 23.2. *There exists a normalized function $\alpha(u)$, of bounded variation on $(0, \infty)$, such that*

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u}$$

converges,

$$\int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, d\alpha(t)$$

diverges,

$$\log (1/x) \left| \int_{0+}^{x} H_{k,t}[f(x)]dt \right| \leqq M, \qquad k \geqq 2;\ 0 < x \leqq 1, \tag{23.8}$$

with M independent of k, and

$$\log (1/x) \int_{0+}^{x} H_{k,t}[f(x)]dt = o(1), \qquad x \to 0, \tag{23.9}$$

for each $k \geqq 2$.

The function $\alpha(u)$ is defined by means of the sequences $\{u_n\}$, $\{u_n'\}$, as in Theorem 5.4; the sequences are now further restricted by the conditions that

$$u_{n+1}' < u_n^2, \qquad u_n' < 2u_n, \qquad n = 1, 2, \cdots,$$

and that

$$\sum_{n=1}^{\infty} (u_n' - u_n) u_n^{-2}$$

converges. For example, we might have

$$u_n = 2^{-2^{n^2}}, \qquad u_n' = (1 + n^{-2}u_n)u_n.$$

The first two statements of the theorem were established in Theorem 5.4.

The proof of the necessity of the conditions of Theorem 23.1 shows that (23.8) is satisfied if

$$\alpha(u) = O(-1/\log u), \qquad u \to 0,$$

(which is readily verified in this case), and that

$$\int_{0+}^{x} H_{k,t}[f(x)]dt = \left(\int_{0+}^{x^{1/2}} + \int_{x^{1/2}}^{\infty}\right) d\alpha(u) \int_0^x F_k(u, t)dt$$
$$= I_1 + I_2,$$

with

$$I_2 = O(x^{1/4}) = o(-1/\log x), \qquad x \to 0.$$

It remains only to show that for $k \geqq 2$,

$$(23.10) \qquad I_1 = o(-1/\log x), \qquad x \to 0.$$

Take $x < u_1^2$. We have $I_1 = Q + R$, where

$$Q = \sum_{n=n_0+1}^{\infty} \frac{-1}{\log u_n} \int_0^x [F_k(u_n, t) - F_k(u_n', t)]dt,$$

and n_0 is determined by the condition $u_{n_0} \leqq x^{1/2} < u_{n_0-1}$;

$$R = \begin{cases} -(2\log u_{n_0})^{-1} \displaystyle\int_0^x F_k(u_{n_0}, t)dt, & x^{1/2} = u_{n_0}, \\ -(\log u_{n_0})^{-1} \displaystyle\int_0^x F_k(u_{n_0}, t)dt, & u_{n_0} < x^{1/2} < u_{n_0}', \\ -(\log u_{n_0})^{-1} \displaystyle\int_0^x [F_k(u_{n_0}, t) - F_k(u_{n_0}', t)]dt, & u_{n_0}' < x^{1/2} < u_{n_0} - 1, \\ -(2\log u_{n_0})^{-1} \left\{ \displaystyle\int_0^x F_k(u_{n_0}, t)dt + \int_0^x [F_k(u_{n_0}, t) - F_k(u_{n_0}', t)]dt \right\}, & \\ & x^{1/2} = u_{n_0}'. \end{cases}$$

Now

$$\frac{\partial}{\partial u} \int_0^x F_k(u, t)dt = -\frac{x}{u} F_k(u, x);*$$

hence

$$\int_0^x [F_k(u_n, t) - F_k(u_n', t)]dt = x(u_n' - u_n)(u_n'')^{-1} F_k(u_n'', x), \qquad u_n' < u_n'' < u_n,$$

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* See p. 32.

so that

$$(23.11) \qquad Q = x \sum_{n=n_0+1}^{\infty} \frac{u_n - u_n'}{u_n'' \log u_n} F_k(u_n'', x).$$

For $n \geqq n_0+1$, we have $u_n' < u_{n_0} \leqq x^{1/2}$; since $u'_{n_0+1} < u_{n_0}^2$, we have $u'_{n_0+1} < x$; and (Lemma 7.4) $F_k(u, x)$ is an increasing function of u for $u \leqq u'_{n_0+1}$, $(k \geqq 2)$. Thus, for $n \geqq n_0+1$,

$$F_k(u_n'', x) \leqq F_k(u_n', x), \qquad k \geqq 2.$$

But

$$\begin{aligned} F_k(u_n', x) &= d_k^2 u_n'^k x^{k-1} \int_0^\infty \frac{s^{2k-1} ds}{(s + u_n')^{2k}(s + x)^{2k}} \\ &\leqq d_k^2 u_n'^k \int_0^\infty \frac{s^{k-2} ds}{(s + u_n')^{2k}} \\ &= \frac{A_k}{u_n'} \leqq \frac{A_k}{u_n}, \qquad n \geqq n_0 + 1, \end{aligned}$$

where A_k depends only on k. The function $-1/\log u$ increases for $0 < u < 1$; hence for $n \geqq n_0+1$, one has

$$\frac{-1}{\log u_n} < \frac{-1}{\log u_{n_0}} \leqq \frac{-1}{\log x^{1/2}}.$$

Relation (23.11) now gives

$$\begin{aligned} 0 < Q &\leqq \frac{2A_k x}{\log x} \sum_{n=n_0+1}^{\infty} \frac{u_n - u_n'}{u_n^2} \leqq \frac{2A_k x}{\log x} \sum_{n=1}^{\infty} \frac{u_n - u_n'}{u_n^2} \\ &= O(-x/\log x) = o(-1/\log x), \qquad x \to 0. \end{aligned}$$

If $u_{n_0} \leqq x^{1/2} \leqq u'_{n_0}$, then

$$\begin{aligned} 0 &< \frac{-1}{\log u_{n_0}} \int_0^x F_k(u_{n_0}, t) dt \\ &= \frac{-2d_k^2 u_{n_0}^k}{\log x} \int_0^x t^{k-1} dt \int_0^\infty \frac{s^{2k-1} ds}{(s + t)^{2k}(s + u_{n_0})^{2k}} \\ &\leqq \frac{-2d_k^2 u_{n_0}^k}{\log x} \int_0^x dt \int_0^\infty \frac{s^{k-2} ds}{(s + u_{n_0})^{2k}} \\ &= \frac{-A_k' x}{u_{n_0} \log x} \leqq \frac{-2A_k' x}{u_{n_0}' \log x} \leqq \frac{-2A_k' x^{1/2}}{\log x}, \end{aligned}$$

where A_k' depends only on k. Therefore,

$$0 < R \leqq \frac{-2A_k' x^{1/2}}{\log x}, \qquad u_{n_0} \leqq x^{1/2} < u_{n_0}'.$$

If, on the other hand, $u_{n_0}' \leqq x^{1/2} < u_{n_0-1}$, then

$$\begin{aligned} 0 &< \frac{-1}{\log u_{n_0}} \int_0^x [F_k(u_{n_0}, t) - F_k(u_{n_0}', t)]dt \\ &= \frac{x(u_{n_0} - u_{n_0}')}{u_{n_0}'' \log u_{n_0}} F_k(u_{n_0}'', x), \qquad u_{n_0} < u_{n_0}'' < u_{n_0}', \\ &\leqq \frac{-A_k'' x}{\log x}, \end{aligned}$$

where A_k'' depends only on k, since

$$F_k(u_{n_0}'', x) \leqq A_k/u_{n_0}'' \leqq A_k/u_{n_0}$$

and $(u_{n_0}' - u_{n_0})u_{n_0}^{-2}$ is the general term of a convergent series. Therefore,

$$0 < R \leqq \frac{-A_k'' x}{\log x} - \frac{A_k' x^{1/2}}{\log x}, \qquad u_{n_0}' \leqq x^{1/2} < u_{n_0} - 1.$$

We have shown that

$$Q + R = o(-1/\log x), \qquad x \to 0,$$

and the construction is complete.

24. **Determining function the integral of a function of class L.** The theorem which we establish is little more than a corollary of Theorem 23.1.

THEOREM 24.1. *A necessary and sufficient condition that $f(x)$ should have the representation*

$$f(x) = \int_{0+}^{\infty} \frac{\log (x/t)}{x - t} \phi(t)dt, \tag{24.1}$$

with $\phi(t)$ of class L on $(0, \infty)$, is that $f(x)$ should satisfy Conditions D *and* (23.2).

The conditions are necessary, by Theorems 19.1 and 23.1, since (24.1) can be written

$$f(x) = \int_{0+}^{\infty} \frac{\log (x/t)}{x - t} d\alpha(t),$$

$$\alpha(t) = \int_{0+}^{t} \phi(u)du, \tag{24.2}$$

$$(24.3) \qquad \int_{0+}^{\infty} |d\alpha(t)| = \int_{0}^{\infty} |\phi(u)|\,du = M < \infty.$$

The conditions are sufficient. By Theorem 19.1

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{\phi(u)du}{t+u}, \qquad \int_{0}^{\infty} |\phi(u)|\,du = M < \infty;$$

this we may write as

$$f(x) = \int_{0+}^{\infty} \frac{dt}{x+t} \int_{0+}^{\infty} \frac{d\alpha(u)}{t+u},$$

where $\alpha(t)$ is defined by (24.2) and has the property (24.3); and we have

$$(24.4) \qquad \alpha(t) = \lim_{k\to\infty} \int_{0+}^{t} H_{k,u}[f(x)]du, \qquad t > 0.$$

By Theorem 17.1

$$\int_{0}^{\infty} |H_{k,t}[f(x)]|\,dt \leqq M, \qquad k = 2, 3, \cdots.$$

Then $f(x)$ satisfies the hypotheses of Theorem 23.1, and

$$(24.5) \qquad f(x) = \int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, d\beta(t),$$

where

$$\beta(t) = \lim_{k\to\infty} \int_{0+}^{t} H_{k,u}[f(x)]du, \qquad t > 0.$$

Comparing this with (24.4), we obtain

$$\beta(t) \equiv \alpha(t) = \int_{0}^{t} \phi(u)du,$$

and (24.5) reduces to (24.1).

25. **Determining function the integral of a bounded function.** We prove the following theorem:

THEOREM 25.1. *A necessary and sufficient condition that $f(x)$ should have the representation*

$$(25.1) \qquad f(x) = \int_{0+}^{\infty} \frac{\log (x/t)}{x-t}\, \phi(t)dt,$$

with $\phi(t)$ bounded almost everywhere, is that $f(x)$ should satisfy Conditions E, *and that for an infinite sequence of integers k,*

$$\log x \left| \int_x^\infty t^{-1} H_{k,t}[f(x)]dt \right| \leqq \epsilon(x), \qquad x \geqq 4, \tag{25.2}$$

where $\lim_{x\to\infty} \epsilon(x) = 0$, *and $\epsilon(x)$ is independent of k.*

To establish the necessity of the conditions, we have only to establish (25.2), because of Theorems 20.1 and 5.2. We have

$$H_{k,t}[f(x)] = \int_0^\infty F_k(u, t)\phi(u)du, \qquad k \geqq 2.$$

Take $x>1$. Then

$$\int_x^\infty t^{-1}H_{k,t}[f(x)]dt = \int_x^\infty t^{-1}dt\left(\int_0^1 + \int_1^\infty\right)F_k(u, t)\phi(u)du$$
$$= I_1 + I_2.$$

To discuss I_2, set

$$\beta(u) = \int_u^\infty t^{-1}\phi(t)dt.$$

By Theorem 3.1,

$$\beta(u) = o(1/\log u), \qquad u \to \infty; \tag{25.3}$$

then, since $F_k(u, t) = O(1/u)$, $(u\to\infty)$, we have

$$\begin{aligned} \int_1^\infty F_k(u, t)\phi(u)du &= -\int_1^\infty uF_k(u, t)d\beta(u) \\ &= \beta(1)F_k(1, t) + \int_1^\infty \beta(u)\frac{\partial}{\partial u}(uF_k(u, t))du. \end{aligned} \tag{25.4}$$

By the homogeneity of $F_k(u, t)$,

$$\frac{\partial}{\partial u}(uF_k(u, t)) = -t\frac{\partial}{\partial t}F_k(u, t);$$

therefore

$$\int_1^\infty \beta(u)\frac{\partial}{\partial u}(uF_k(u, t))du = -t\int_1^\infty \beta(u)\frac{\partial}{\partial t}F_k(u, t)du. \tag{25.5}$$

It is easily verified that, for each k, the integral on the right is uniformly convergent for $t \geqq \delta > 0$, and hence that we may take the symbol $\partial/\partial t$ outside the integral sign. Using (25.5) in (25.4), then, we obtain

$$I_2 = \beta(1)\int_x^\infty t^{-1}F_k(1, t)dt - \int_1^\infty \beta(u)F_k(u, t)du \Big|_{t=x}^{\infty} = I_2' + I_2''. \tag{25.6}$$

Consider the integral

$$\int_1^\infty \beta(u)F_k(u, x)du = \left(\int_1^{x^{1/2}} + \int_{x^{1/2}}^\infty\right)\beta(u)F_k(u, x)du. \tag{25.7}$$

$$\begin{aligned} \left|\int_{x^{1/2}}^\infty \beta(u)F_k(u, x)du\right| &\leqq \underset{u\geqq x^{1/2}}{\text{u.b.}} \,|\beta(u)| \int_{x^{1/2}}^\infty F_k(u, x)du \\ &\leqq \underset{u\geqq x^{1/2}}{\text{u.b.}} \,|\beta(u)|. \end{aligned} \tag{25.8}$$

There is a constant B such that $|\beta(u)| \leqq B$, $(1 \leqq u < \infty)$. Then

$$\begin{aligned} \left|\int_1^{x^{1/2}} \beta(u)F_k(u, x)du\right| &\leqq B\int_1^{x^{1/2}} F_k(u, x)du \\ &= Bx\int_{x^{-1}}^{x^{-1/2}} F_k(vx, x)dv, \qquad u = vx, \\ &= B\int_{x^{-1}}^{x^{-1/2}} F_k(v, 1)dv \\ &= B\int_{x^{-1}}^{x^{-1/2}} vF_k(1, v)dv \\ &\leqq B\int_0^{x^{-1/2}} F_k(1, v)dv, \qquad x > 1, \\ &= BH_k(x^{-1/2}). \end{aligned}$$

Applying Lemma 22.1, we then have

$$\left|\int_1^{x^{1/2}} \beta(u)F_k(u, x)du\right| \leqq ABx^{-1/4}, \qquad x \geqq 4. \tag{25.9}$$

Now

$$\beta(u) = o(1/\log u), \qquad u \to \infty.$$

Combining (25.9), (25.8), and (25.3) and referring to (25.7), (25.6), we see that for $x \geqq 4$

$$|I_2''| = \left|\int_1^\infty \beta(u)F_k(u, x)du\right| \leqq \epsilon_1(x)/\log x, \tag{25.10}$$

where $\epsilon_1(x) = o(1)$, $(x \to \infty)$, and $\epsilon_1(x)$ does not depend on k.

We have still to discuss

$$I_1 = \int_x^\infty t^{-1}dt \int_0^1 F_k(u, t)\phi(u)du$$

and

$$I_2' = \beta(1)\int_x^\infty t^{-1}F_k(1, t)dt.$$

For $t>u$, $F_k(u, t)$ is an increasing function of u (Lemma 7.4); therefore for $t>1$,

$$\left|\int_0^1 F_k(u, t)\phi(u)du\right| \leqq MF_k(1, t),$$

where $|\phi(u)| \leqq M$ almost everywhere. Hence

$$|I_1| \leqq M\int_x^\infty t^{-1}F_k(1, t)dt, \qquad x > 1,$$

and

$$|I_1 + I_2'| \leqq (M + |\beta(1)|)\int_x^\infty t^{-1}F_k(1, t)dt.$$

But for $x>1$,

$$\begin{aligned}\int_x^\infty t^{-1}F_k(1, t)dt &= \int_0^{x^{-1}} s^{-1}F_k(1, s^{-1})ds, \qquad st = 1,\\ &= \int_0^{x^{-1}} F_k(s, 1)ds = \int_0^{x^{-1}} sF_k(1, s)ds\\ &\leqq \int_0^{x^{-1}} F_k(1, s)ds = H_k(x^{-1}).\end{aligned}$$

Again by Lemma 22.1, we see that

$$|I_1 + I_2'| \leqq \epsilon_2(x)/\log x, \qquad x \geqq 4,$$

where $\epsilon_2(x) = o(1)$, $(x\to\infty)$, and is independent of k. Combining this with (25.10), we have (25.2).

We now establish the sufficiency of our conditions. By Theorem 20.1, Conditions E imply that

$$f(x) = \int_{0+}^\infty \frac{dt}{x+t}\int_{0+}^\infty \frac{\phi(u)du}{t+u}, \tag{25.11}$$

with $|\phi(u)| \leqq M$ almost everywhere. By Theorem 10.1

$$\phi(u) = \lim_{k\to\infty} H_{k,u}[f(x)] \tag{25.12}$$

for almost all u. Since (25.2) is satisfied, we have

$$\left| \int_x^\infty t^{-1} H_{k,t}[f(x)]dt \right| \leqq \epsilon(x)/\log x, \qquad x \geqq 4,$$

for an infinite sequence of integers k, with $\epsilon(x)$ independent of k, and $\epsilon(x) = o(1)$, $(k \to \infty)$. Then for k in the sequence,

$$\left| \int_{x_1}^{x_2} \frac{1}{t} H_{k,t}[f(x)]dt \right| \leqq \frac{\epsilon(x_1)}{\log x_1} + \frac{\epsilon(x_2)}{\log x_2}, \qquad x_2 \geqq x_1 \geqq 4.$$

Let $k \to \infty$ in the sequence. By (25.12) and bounded convergence,

$$\left| \int_{x_1}^{x_2} \frac{\phi(t)}{t} dt \right| \leqq \frac{\epsilon(x_1)}{\log x_1} + \frac{\epsilon(x_2)}{\log x_2}, \qquad x_2 \geqq x_1 \geqq 4.$$

Therefore

$$\int_1^\infty t^{-1}\phi(t)dt$$

converges, and, if we set $x_1 = x$ and let $x_2 \to \infty$, we obtain

$$(25.13) \qquad \left| \int_x^\infty \frac{\phi(t)}{t} dt \right| \leqq \frac{\epsilon(x)}{\log x} = o\left(\frac{1}{\log x}\right), \qquad x \to \infty.$$

Moreover,

$$(25.14) \qquad \left| \int_0^t \phi(u)du \right| \leqq Mt = o(-1/\log t), \qquad t \to 0.$$

Relations (25.13) and (25.14) are the conditions of Theorem 5.3; this theorem now permits us to change the order of integration in (25.11), obtaining the representation (25.1) for $f(x)$.

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